

QUADRI-BIALGEBRAS

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ABSTRACT. We introduce a notion of quadri-bialgebra, which gives a bialgebra theory built upon a quadri-algebra introduced by Aguiar and Loday. A quadri-bialgebra is equivalent to a Manin triple of dendriform dialgebras associated to a nondegenerate 2-cocycle or a Manin triple of quadri-algebras associated to a nondegenerate invariant bilinear form. Moreover, quadri-bialgebras fit into a framework of construction of certain linear operators on the double spaces.

1. INTRODUCTION

At the beginning of 1990s, in order to study periodicity phenomena in algebraic K-theory, J.-L. Loday introduced the notion of dendriform dialgebra ([Lo1]). It has been attracting a great interest because of its connections with various fields in mathematics and physics (see [EMP] and the references therein).

There is a remarkable fact that a Rota-Baxter operator (of weight zero), which first arose in probability theory ([Bax]) and later became a subject in combinatorics ([R1, R2]), on an associative algebra naturally gives a dendriform dialgebra structure on the underlying vector space of the associative algebra ([Ag1, Ag2, E1]). Such unexpected relationships between dendriform dialgebras in the field of operads and algebraic topology and Rota-Baxter operators in the field of combinatorics and probability theory attracted many mathematicians' attentions right away. Later, in 2003, in order to determine the algebraic structures behind a pair of commuting Rota-Baxter operators (on an associative algebra), which appear, for example, in the space of the linear endomorphisms of an infinitesimal bialgebra, Aguiar and Loday introduced the notion of quadri-algebra ([AL]), which is a vector space equipped with four operations satisfying nine axioms. Moreover, quadri-algebras have deep relationships with combinatorics and the theory of Hopf algebras ([AL, EG1]). A quadri-algebra is also regarded as the underlying algebra structure of a dendriform algebra with a nondegenerate 2-cocycle ([Bai3]).

In this paper, we construct a bialgebra theory of quadri-algebras. We are mainly motivated by the theory of Lie bialgebras ([CP, D]). Explicitly, in the finite-dimensional case, we consider an analogue of Manin triple of Lie algebras which is equivalent to a Lie bialgebra, namely, a Manin triple of dendriform dialgebras associated to a nondegenerate 2-cocycle. We find that it is in fact equivalent to certain bialgebra structure of the underlying quadri-algebra, which leads

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to the notion of quadri-bialgebra. Furthermore, it is interesting to show that such a structure is also equivalent to another Manin triple at the level of quadri-algebras, that is, a Manin triple of quadri-algebras with a nondegenerate invariant bilinear form. Quadri-bialgebras have certain similar properties of Lie bialgebras. For example, there are the so-called coboundary quadri-bialgebras which lead to a construction from an analogue of the classical Yang-Baxter equation and there also exists a “Drinfeld double” construction for a finite-dimensional quadri-bialgebra.

Moreover, we find that quadri-bialgebras fit into a framework of construction of certain linear operators, such as Rota-Baxter operators and Nijenhuis operators in combinatorics ([Bax, E1, E2, R1, R2]), renormalization of perturbative quantum field theory (pQFT) ([CK, EG2, EGK1, EGK2]) and quantum physics ([CGM]), on the “double spaces”. We would like to point out that quadri-bialgebras might be put into the framework of the so-called generalized bialgebras in the sense of Loday ([Lo2]), which will be considered elsewhere.

The paper is organized as follows. In Section 2, we recall some basic facts on dendriform dialgebras and quadri-algebras. In Section 3, we introduce the notion of Manin triple of dendriform dialgebras associated to a nondegenerate 2-cocycle and then interpret it in terms of matched pairs of dendriform dialgebras. In Section 4, we introduce the notion of Manin triple of quadri-algebras associated to a nondegenerate invariant bilinear form and then interpret it in terms of matched pairs of quadri-algebras. We also show the equivalence between these two Manin triples. In Section 5, we introduce the notion of quadri-bialgebra as an equivalent bialgebra structure corresponding to the aforementioned Manin triples by assuming that there is a quadri-algebra structure on the dual space. In Section 6, we study the coboundary cases which lead to a construction from certain algebraic equations, which could be regarded as analogues of the classical Yang-Baxter equation. In Section 7, we recall some results in [Bai3] which reduce the aforementioned algebraic equations in a simple form, namely, Q -equation referring to a set of two equations. We list some properties of Q -equation including the ones given in [Bai3] from another point of view. In Section 8, we construct families of Nijenhuis operators and Rota-Baxter operators on certain “double spaces” of quadri-algebras, including the Drinfeld Q -doubles obtained from quadri-bialgebras.

Throughout this paper, all algebras and vector spaces are finite-dimensional over a fixed base field \mathbb{F} . We give some notations as follows.

(1) Let V be a vector space. Let $\mathfrak{B} : V \otimes V \rightarrow \mathbb{F}$ be a symmetric or skew-symmetric bilinear form on a vector space V . A subspace W is called *isotropic* if $W \subset W^\perp$, where

$$(1.1) \quad W^\perp = \{x \in V \mid \mathfrak{B}(x, y) = 0, \forall y \in W\}.$$

(2) Let (A, \diamond) be a vector space with a bilinear operation $\diamond : A \otimes A \rightarrow A$. Let $L_\diamond(x)$ and $R_\diamond(x)$ denote the left and right multiplication operator respectively, that is, $L_\diamond(x)y = R_\diamond(y)x = x \diamond y$

for any $x, y \in A$. We also simply denote them by $L(x)$ and $R(x)$ respectively without confusion. Moreover, let $L_\diamond, R_\diamond : A \rightarrow \mathfrak{gl}(A)$ be two linear maps with $x \rightarrow L_\diamond(x)$ and $x \rightarrow R_\diamond(x)$ respectively.

(3) Let V be a vector space and let $r = \sum_i a_i \otimes b_i \in V \otimes V$. Set

$$(1.2) \quad r_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i,$$

where 1 is a symbol playing a similar role of unit. If in addition, there exists a bilinear operation $\diamond : V \otimes V \rightarrow V$ on V , then the operation between two r s is in an obvious way. For example,

$$(1.3) \quad r_{12} \diamond r_{13} = \sum_{i,j} a_i \diamond a_j \otimes b_i \otimes b_j, \quad r_{13} \diamond r_{23} = \sum_{i,j} a_i \otimes a_j \otimes b_i \diamond b_j, \quad r_{23} \diamond r_{12} = \sum_{i,j} a_j \otimes a_i \diamond b_j \otimes b_i.$$

and so on. Note Eq. (1.3) is independent of the existence of the unit.

(4) Let V be a vector space. Any $r \in V \otimes V$ can be identified as a linear map $T_r : V^* \rightarrow V$ in the following way:

$$(1.4) \quad \langle u^* \otimes v^*, r \rangle = \langle u^*, T_r(v^*) \rangle, \quad \forall u^*, v^* \in V^*,$$

where \langle, \rangle is the canonical pairing between V and V^* . $r \in V \otimes V$ is called *nondegenerate* if the above induced linear map T_r is invertible.

(5) Let V_1, V_2 be two vector spaces and let $T : V_1 \rightarrow V_2$ be a linear map. Denote the dual (linear) map by $T^* : V_2^* \rightarrow V_1^*$ defined by

$$(1.5) \quad \langle v_1, T^*(v_2^*) \rangle = \langle T(v_1), v_2^* \rangle, \quad \forall v_1 \in V_1, v_2^* \in V_2^*.$$

(6) Let A be an algebra and let V be a vector space. For any linear map $\rho : A \rightarrow \mathfrak{gl}(V)$, define a linear map $\rho^* : A \rightarrow \mathfrak{gl}(V^*)$ by

$$(1.6) \quad \langle \rho^*(x)v^*, u \rangle = \langle v^*, \rho(x)u \rangle, \quad \forall x \in A, u \in V, v^* \in V^*.$$

Note that in this case, ρ^* is different from the one given by Eq. (1.5) which regards $\mathfrak{gl}(V)$ as a vector space, too.

(7) Let V be a vector space, we sometimes use 1 to denote the identity transformation of V .

2. DENDRIFORM DIALGEBRAS AND QUADRI-ALGEBRAS

Definition 2.1. ([Lo1]) A *dendriform dialgebra* (A, \prec, \succ) is a vector space A together with two bilinear operations $\prec, \succ : A \otimes A \rightarrow A$ such that (for any $x, y, z \in A$)

$$(2.1) \quad (x \prec y) \prec z = x \prec (y \star z), (x \succ y) \prec z = x \succ (y \prec z), (x \star y) \succ z = x \succ (y \succ z),$$

where $x \star y = x \prec y + x \succ y$. Moreover, a *homomorphism* between two dendriform dialgebras is defined as a linear map (between the two dendriform dialgebras) which preserves the operations respectively.

Remark 2.2. ([Lo1]) For a dendriform dialgebra (A, \prec, \succ) , the bilinear operation given by

$$(2.2) \quad x \star y := x \prec y + x \succ y, \quad \forall x, y \in A,$$

defines an associative algebra, which is denoted by $(As(A), \star)$.

Definition 2.3. ([Ag3, Bai2]) Let (A, \prec, \succ) be a dendriform dialgebra and let V be a vector space. Let $l_{\prec}, l_{\succ}, r_{\prec}, r_{\succ} : A \rightarrow \mathfrak{gl}(V)$ be four linear maps. V or $(V, l_{\prec}, r_{\prec}, l_{\succ}, r_{\succ})$ is called a *bimodule* of A if and only if the following equations holds (for any $x, y \in A$):

$$(2.3) \quad r_{\prec}(y)r_{\prec}(x) = r_{\prec}(x \star y), r_{\prec}(y)l_{\prec}(x) = l_{\prec}(x)r_{\prec}(y), l_{\prec}(x \prec y) = l_{\prec}(x)l_{\prec}(y),$$

$$(2.4) \quad r_{\prec}(y)r_{\succ}(x) = r_{\succ}(x \prec y), r_{\prec}(y)l_{\succ}(x) = l_{\succ}(x)r_{\prec}(y), l_{\prec}(x \succ y) = l_{\succ}(x)l_{\prec}(y),$$

$$(2.5) \quad r_{\succ}(y)r_{\star}(x) = r_{\succ}(x \succ y), r_{\succ}(y)l_{\star}(x) = l_{\succ}(x)r_{\succ}(y), l_{\succ}(x \star y) = l_{\succ}(x)l_{\succ}(y).$$

Proposition 2.4. ([Bai2]) Let $(V, l_{\prec}, r_{\prec}, l_{\succ}, r_{\succ})$ be a bimodule of a dendriform dialgebra (A, \prec, \succ) . Then $(V^*, -r_{\succ}^*, l_{\succ}^* + l_{\prec}^*, r_{\prec}^* + r_{\succ}^*, -l_{\prec}^*)$ is a bimodule of (A, \prec, \succ) .

Definition 2.5. ([Bai2, K]) Let $(V, l_{\prec}, r_{\prec}, l_{\succ}, r_{\succ})$ be a bimodule of a dendriform dialgebra (A, \prec, \succ) . A linear map $T : V \rightarrow A$ is called an \mathcal{O} -operator associated to $(V, l_{\prec}, r_{\prec}, l_{\succ}, r_{\succ})$ if T satisfies

$$(2.6) \quad T(u) \prec T(v) = T(l_{\prec}(T(u))v + r_{\prec}(T(v))u), T(u) \succ T(v) = T(l_{\succ}(T(u))v + r_{\succ}(T(v))u), \forall u, v \in V.$$

Definition 2.6. ([AL]) A *quadri-algebra* $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ is a vector space A together with four bilinear operations $\nwarrow, \nearrow, \swarrow$ and $\searrow : A \otimes A \rightarrow A$ satisfying the axioms below (for any $x, y, z \in A$)

$$(2.7) \quad (x \nwarrow y) \nwarrow z = x \nwarrow (y \star z), (x \nearrow y) \nwarrow z = x \nearrow (y \prec z), (x \wedge y) \nearrow z = x \nearrow (y \succ z),$$

$$(2.8) \quad (x \swarrow y) \nwarrow z = x \swarrow (y \wedge z), (x \searrow y) \nwarrow z = x \searrow (y \nwarrow z), (x \vee y) \nearrow z = x \searrow (y \nearrow z),$$

$$(2.9) \quad (x \prec y) \swarrow z = x \swarrow (y \vee z), (x \succ y) \swarrow z = x \searrow (y \swarrow z), (x \star y) \searrow z = x \searrow (y \searrow z),$$

where

$$(2.10) \quad x \succ y := x \nearrow y + x \searrow y, x \prec y := x \nwarrow y + x \swarrow y,$$

$$(2.11) \quad x \vee y := x \swarrow y + x \searrow y, x \wedge y := x \nwarrow y + x \nearrow y,$$

$$(2.12) \quad x \star y := x \searrow y + x \nearrow y + x \nwarrow y + x \swarrow y = x \succ y + x \prec y = x \vee y + x \wedge y.$$

A *homomorphism* between two quadri-algebras is defined as a linear map (between the two quadri-algebras) which preserves the operations respectively.

Proposition 2.7. ([AL]) Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra.

(1) The bilinear operation given by Eq. (2.10) defines a dendriform dialgebra $A_h := (A, \prec, \succ)$, which is called the associated horizontal dendriform dialgebra.

(2) The bilinear operation given by Eq. (2.11) defines a dendriform dialgebra $A_v := (A, \wedge, \vee)$, which is called the associated vertical dendriform dialgebra.

Proposition 2.8. ([Bai3]) *Let A be a vector space with four bilinear operations denoted by $\nwarrow, \nearrow, \swarrow$ and $\searrow: A \otimes A \rightarrow A$. Then the following conditions are equivalent:*

- (1) $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ is a quadri-algebra;
- (2) (A, \prec, \succ) defined by Eq. (2.10) is a dendriform dialgebra and $(A, L_{\swarrow}, R_{\nwarrow}, L_{\searrow}, R_{\nearrow})$ is a bimodule.
- (3) (A, \wedge, \vee) defined by Eq. (2.11) is a dendriform dialgebra and $(A, L_{\nearrow}, R_{\nwarrow}, L_{\searrow}, R_{\swarrow})$ is a bimodule.

Corollary 2.9. ([Bai3]) *Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra. Then $(A^*, -R_{\nearrow}^*, L_{\vee}^*, R_{\wedge}^*, -L_{\swarrow}^*)$ is a bimodule of the associated horizontal dendriform dialgebra (A, \prec, \succ) and $(A^*, -R_{\swarrow}^*, L_{\prec}^*, R_{\succ}^*, -L_{\nearrow}^*)$ is a bimodule of the associated vertical dendriform dialgebra (A, \wedge, \vee) .*

For brevity, we pay our main attention to the study of the associated vertical dendriform dialgebras of quadri-algebras in this paper. The corresponding study on the associated horizontal dendriform dialgebras is completely similar.

Definition 2.10. ([Bai2]) Let (A, \wedge, \vee) be a dendriform dialgebra and let $(As(A), \star)$ be the associated associative algebra. Suppose that $\mathfrak{B}: A \otimes A \rightarrow \mathbb{F}$ is a symmetric bilinear form. \mathfrak{B} is called a 2-cocycle of A if \mathfrak{B} satisfies

$$(2.13) \quad \mathfrak{B}(x \star y, z) = \mathfrak{B}(y, z \wedge x) + \mathfrak{B}(x, y \vee z), \forall x, y, z \in A.$$

Proposition 2.11. ([Bai3]) *Let (A, \wedge, \vee) be a dendriform dialgebra equipped with a nondegenerate 2-cocycle \mathfrak{B} . Define four bilinear operations $\nwarrow, \nearrow, \swarrow, \searrow: A \otimes A \rightarrow A$ by*

$$(2.14) \quad \mathfrak{B}(x \nwarrow y, z) = \mathfrak{B}(x, y \star z), \mathfrak{B}(x \nearrow y, z) = -\mathfrak{B}(y, z \vee x),$$

$$(2.15) \quad \mathfrak{B}(x \swarrow y, z) = -\mathfrak{B}(x, y \wedge z), \mathfrak{B}(x \searrow y, z) = \mathfrak{B}(y, z \star x), \forall x, y, z \in A.$$

Then $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ is a quadri-algebra such that (A, \wedge, \vee) is the associated vertical dendriform dialgebra.

3. MANIN TRIPLES OF DENDRIFORM DIALGEBRAS ASSOCIATED TO A NONDEGENERATE 2-COCYCLE AND MATCHED PAIRS OF DENDRIFORM DIALGEBRAS

Definition 3.1. A Manin triple of dendriform dialgebras associated to a nondegenerate 2-cocycle is a triple of dendriform dialgebras (A, A^+, A^-) together with a nondegenerate 2-cocycle \mathfrak{B} on A , such that:

- (1) A^+ and A^- are sub-dialgebras of A ;
- (2) $A = A^+ \oplus A^-$ as vector spaces;
- (3) A^+ and A^- are isotropic with respect to \mathfrak{B} .

It is denoted by $(A, A^+, A^-, \mathfrak{B})$. A *homomorphism* between two Manin triples of dendriform dialgebras associated to a nondegenerate 2-cocycle $(A, A^+, A^-, \mathfrak{B}_A)$ and $(B, B^+, B^-, \mathfrak{B}_B)$ is a homomorphism of dendriform dialgebras $\varphi : A \rightarrow B$ such that

$$(3.1) \quad \varphi(A^+) \subset B^+, \varphi(A^-) \subset B^-, \mathfrak{B}_A(x, y) = \mathfrak{B}_B(\varphi(x), \varphi(y)), \forall x, y \in A.$$

Definition 3.2. Let (A, \wedge, \vee) be a dendriform dialgebra. If there is a dendriform dialgebra structure on the direct sum of the underlying vector spaces of A and the dual space A^* such that A and A^* are sub-dialgebras and the natural symmetric bilinear form on $A \oplus A^*$ given by

$$(3.2) \quad \mathfrak{B}_S(x + a^*, y + b^*) := \langle a^*, y \rangle + \langle x, b^* \rangle, \forall x, y \in A; a^*, b^* \in A^*,$$

is a 2-cocycle, then $(A \oplus A^*, A, A^*, \mathfrak{B}_S)$ is called a *standard Manin triple of dendriform dialgebras associated to \mathfrak{B}_S* .

Obviously, a standard Manin triple of dendriform dialgebras is a Manin triple of dendriform dialgebras. Conversely, we have

Proposition 3.3. *Every Manin triple of dendriform dialgebras is isomorphic to a standard one.*

Proof. Since in this case A^- and $(A^+)^*$ are identified by the nondegenerate 2-cocycle, the dendriform dialgebra structure on A^- is transferred to $(A^+)^*$. Hence the dendriform dialgebra structure on $A^+ \oplus A^-$ is transferred to $A^+ \oplus (A^+)^*$. Then the conclusion holds. \square

Proposition 3.4. ([Bai2]) *Let (A, \wedge_A, \vee_A) and (B, \wedge_B, \vee_B) be two dendriform dialgebras. Suppose that there are linear maps $l_{\wedge_A}, r_{\wedge_A}, l_{\vee_A}, r_{\vee_A} : A \rightarrow \mathfrak{gl}(B)$ and $l_{\wedge_B}, r_{\wedge_B}, l_{\vee_B}, r_{\vee_B} : B \rightarrow \mathfrak{gl}(A)$ such that $(l_{\wedge_A}, r_{\wedge_A}, l_{\vee_A}, r_{\vee_A})$ is a bimodule of A and $(l_{\wedge_B}, r_{\wedge_B}, l_{\vee_B}, r_{\vee_B})$ is a bimodule of B , and they satisfy the following conditions:*

$$(3.3) \quad (l_{\wedge_B}(a)x) \wedge_A y + l_{\wedge_B}(r_{\wedge_A}(x)a)y = l_{\wedge_B}(a)(x \star_A y),$$

$$(3.4) \quad l_{\wedge_B}(l_{\wedge_A}(x)a)y + (r_{\wedge_B}(a)x) \wedge_A y = x \wedge_A (l_{\star_B}(a)y) + r_{\wedge_B}(r_{\star_A}(y)a)x,$$

$$(3.5) \quad r_{\wedge_B}(a)(x \wedge_A y) = r_{\wedge_B}(l_{\star_A}(y)a)x + x \wedge_A (r_{\star_B}(a)y),$$

$$(3.6) \quad (l_{\vee_B}(a)x) \wedge_A y + l_{\vee_B}(r_{\vee_A}(x)a)y = l_{\vee_B}(a)(x \vee_A y),$$

$$(3.7) \quad l_{\vee_B}(l_{\vee_A}(x)a)y + (r_{\vee_B}(a)x) \wedge_A y = x \vee_A (l_{\wedge_B}(a)y) + r_{\vee_B}(r_{\wedge_A}(y)a)x,$$

$$(3.8) \quad r_{\vee_B}(a)(x \vee_A y) = r_{\vee_B}(l_{\wedge_A}(y)a)x + x \vee_A (r_{\wedge_B}(a)y),$$

$$(3.9) \quad (l_{\star_B}(a)x) \vee_A y + l_{\vee_B}(r_{\star_A}(x)a)y = l_{\vee_B}(a)(x \vee_A y),$$

$$(3.10) \quad l_{\vee_B}(l_{\star_A}(x)a)y + (r_{\star_B}(a)x) \vee_A y = x \vee_A (l_{\vee_B}(a)y) + r_{\vee_B}(r_{\vee_A}(y)a)x,$$

$$(3.11) \quad r_{\vee_B}(a)(x \star_A y) = r_{\vee_B}(l_{\vee_A}(y)a)x + x \vee_A (r_{\vee_B}(a)y),$$

$$(3.12) \quad (l_{\wedge_A}(x)a) \wedge_B b + l_{\wedge_A}(r_{\wedge_B}(a)x)b = l_{\wedge_A}(x)(a \star_B b),$$

$$(3.13) \quad l_{\wedge_A}(l_{\wedge_B}(a)x)b + (r_{\wedge_A}(x)a) \wedge_B b = a \wedge_B (l_{\star_A}(x)b) + r_{\wedge_A}(r_{\star_B}(b)x)a,$$

$$(3.14) \quad r_{\wedge_A}(x)(a \wedge_B b) = r_{\wedge_A}(l_{\star_B}(b)x)a + a \wedge_B (r_{\star_A}(x)b),$$

$$(3.15) \quad (l_{\vee_A}(x)a) \wedge_B b + l_{\wedge_A}(r_{\vee_B}(a)x)b = l_{\vee_A}(x)(a \wedge_B b),$$

$$(3.16) \quad l_{\wedge_A}(l_{\vee_B}(a)x)b + (r_{\vee_A}(x)a) \wedge_B b = a \vee_B (l_{\wedge_A}(x)b) + r_{\vee_A}(r_{\wedge_B}(b)x)a,$$

$$(3.17) \quad r_{\wedge_A}(x)(a \vee_B b) = r_{\vee_A}(l_{\wedge_B}(b)x)a + a \vee_B (r_{\wedge_A}(x)b),$$

$$(3.18) \quad (l_{\star_A}(x)a) \vee_B b + l_{\vee_A}(r_{\star_B}(a)x)b = l_{\vee_A}(x)(a \vee_B b),$$

$$(3.19) \quad l_{\vee_A}(l_{\star_B}(a)x)b + (r_{\star_A}(x)a) \vee_B b = a \vee_B (l_{\vee_A}(x)b) + r_{\vee_A}(r_{\vee_B}(b)x)a,$$

$$(3.20) \quad r_{\vee_A}(x)(a \star_B b) = r_{\vee_A}(l_{\vee_B}(b)x)a + a \vee_B (r_{\vee_A}(x)b),$$

for all $x, y \in A, a, b \in B$. Then there is a dendriform dialgebra structure on the vector space $A \oplus B$ which is given by

$$(3.21) \quad (x + a) \wedge (y + b) := x \wedge_A y + l_{\wedge_B}(a)y + r_{\wedge_B}(b)x + a \wedge_B b + l_{\wedge_A}(x)b + r_{\wedge_A}(y)a,$$

$$(3.22) \quad (x + a) \vee (y + b) := x \vee_A y + l_{\vee_B}(a)y + r_{\vee_B}(b)x + a \vee_B b + l_{\vee_A}(x)b + r_{\vee_A}(y)a,$$

for all $x, y \in A, a, b \in B$. We denote this dendriform dialgebra by $A \bowtie_{l_{\wedge_B}, r_{\wedge_B}, l_{\vee_B}, r_{\vee_B}}^{l_{\wedge_A}, r_{\wedge_A}, l_{\vee_A}, r_{\vee_A}} B$ or simply $A \bowtie B$. Moreover $(A, B, l_{\wedge_A}, r_{\wedge_A}, l_{\vee_A}, r_{\vee_A}, l_{\wedge_B}, r_{\wedge_B}, l_{\vee_B}, r_{\vee_B})$ is called a matched pair of dendriform dialgebras. On the other hand, every dendriform dialgebra which is the direct sum of the underlying vector spaces of two sub-dialgebras can be obtained from the above way.

Proposition 3.5. Let $(A, \nwarrow_A, \nearrow_A, \swarrow_A, \searrow_A)$ be a quadri-algebra. Suppose that there is a quadri-algebra structure $\nwarrow_B, \nearrow_B, \swarrow_B, \searrow_B$ on the dual space A^* . Then there exists a dendriform dialgebra structure on the vector space $A \oplus A^*$ such that A_v and $(A^*)_v$ are isotropic sub-dialgebras associated to the 2-cocycle (3.2), that is, $(A_v \oplus (A^*)_v, A_v, (A^*)_v, \mathfrak{B}_S)$ is a Manin triple of dendriform dialgebras, if and only if $(A_v, (A^*)_v, -R_{\swarrow_A}^*, L_{\searrow_A}^*, R_{\nwarrow_A}^*, -L_{\nearrow_A}^*, -R_{\swarrow_B}^*, L_{\searrow_B}^*, R_{\nwarrow_B}^*, -L_{\nearrow_B}^*)$ is a matched pair of dendriform dialgebras.

Proof. If $(A_v, (A^*)_v, -R_{\swarrow_A}^*, L_{\searrow_A}^*, R_{\nwarrow_A}^*, -L_{\nearrow_A}^*, -R_{\swarrow_B}^*, L_{\searrow_B}^*, R_{\nwarrow_B}^*, -L_{\nearrow_B}^*)$ is a matched pair of dendriform dialgebras, then it is straightforward to check that the bilinear form given by Eq. (3.2) on $A_v \bowtie_{-R_{\swarrow_A}^*, L_{\searrow_A}^*, R_{\nwarrow_A}^*, -L_{\nearrow_A}^*}^{-R_{\swarrow_B}^*, L_{\searrow_B}^*, R_{\nwarrow_B}^*, -L_{\nearrow_B}^*} (A^*)_v$ is a 2-cocycle. So $(A_v \oplus (A_v)^*, A_v, (A^*)_v, \mathfrak{B}_S)$ is a Manin triple of dendriform dialgebras.

Conversely, if $(A_v \oplus (A_v)^*, A_v, (A_v)^*, \mathfrak{B}_S)$ is a Manin triple of dendriform dialgebras, then for all $x, y \in A, a^*, b^* \in A^*$, we have

$$\langle x \wedge a^*, y \rangle = \mathfrak{B}_S(y, x \wedge a^*) = -\mathfrak{B}_S(y \swarrow_A x, a^*) = \langle y, -R_{\swarrow_A}^*(x)a^* \rangle,$$

$$\begin{aligned}\langle x \star a^*, b^* \rangle &= \mathfrak{B}_S(b^*, x \star a^*) = \mathfrak{B}_S(a^* \searrow_B b^*, x) = \langle L_{\searrow_B}^*(a^*)x, b^* \rangle, \\ \langle x \vee a^*, b^* \rangle &= \mathfrak{B}_S(b^*, x \vee a^*) = -\mathfrak{B}_S(a^* \nearrow_B b^*, x) = \langle -L_{\nearrow_B}^*(a^*)x, b^* \rangle.\end{aligned}$$

So $x \wedge a^* = -R_{\swarrow_A}^*(x)a^* + L_{\searrow_B}^*(a^*)x$. Similarly, we show that

$x \vee a^* = R_{\swarrow_A}^*(x)a^* - L_{\nearrow_B}^*(a^*)x$, $a^* \wedge x = -R_{\swarrow_B}^*(a^*)x + L_{\searrow_A}^*(x)a^*$, $a^* \vee x = R_{\swarrow_B}^*(a^*)x - L_{\nearrow_A}^*(x)a^*$, for any $x \in A, a^* \in A^*$. Hence $(A_v, (A^*)_v, -R_{\swarrow_A}^*, L_{\searrow_A}^*, R_{\swarrow_A}^*, -L_{\nearrow_A}^*, -R_{\swarrow_B}^*, L_{\searrow_B}^*, R_{\swarrow_B}^*, -L_{\nearrow_B}^*)$ is a matched pair of dendriform dialgebras. \square

4. BIMODULES AND MATCHED PAIRS OF QUADRI-ALGEBRAS

Definition 4.1. ([Bai3]) Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra and let V be a vector space. Let $l_\circ, r_\circ : A \rightarrow \mathfrak{gl}(V)$ be eight linear maps, where $\circ \in \{\nwarrow, \nearrow, \swarrow, \searrow\}$. V or $(V, l_\nwarrow, r_\nwarrow, l_\nearrow, r_\nearrow, l_\swarrow, r_\swarrow, l_\searrow, r_\searrow)$ is called a *bimodule* of A if for all $x, y \in A$,

$$(4.1) \quad r_\nwarrow(y)r_\nwarrow(x) = r_\nwarrow(x \star y), r_\nwarrow(y)l_\nwarrow(x) = l_\nwarrow(x)r_\star(y), l_\nwarrow(x \nwarrow y) = l_\nwarrow(x)l_\star(y),$$

$$(4.2) \quad r_\nwarrow(y)r_\nearrow(x) = r_\nearrow(x \prec y), r_\nwarrow(y)l_\nearrow(x) = l_\nearrow(x)r_\prec(y), l_\nwarrow(x \nearrow y) = l_\nearrow(x)l_\prec(y),$$

$$(4.3) \quad r_\nearrow(y)r_\wedge(x) = r_\nearrow(x \succ y), r_\nearrow(y)l_\wedge(x) = l_\nearrow(x)r_\succ(y), l_\nearrow(x \wedge y) = l_\nearrow(x)l_\succ(y),$$

$$(4.4) \quad r_\nwarrow(y)r_\swarrow(x) = r_\swarrow(x \wedge y), r_\nwarrow(y)l_\swarrow(x) = l_\swarrow(x)r_\wedge(y), l_\nwarrow(x \swarrow y) = l_\swarrow(x)l_\wedge(y),$$

$$(4.5) \quad r_\nwarrow(y)r_\searrow(x) = r_\searrow(x \nwarrow y), r_\nwarrow(y)l_\searrow(x) = l_\searrow(x)r_\nwarrow(y), l_\nwarrow(x \searrow y) = l_\searrow(x)l_\nwarrow(y),$$

$$(4.6) \quad r_\nearrow(y)r_\vee(x) = r_\searrow(x \nearrow y), r_\nearrow(y)l_\vee(x) = l_\searrow(x)r_\nearrow(y), l_\nearrow(x \vee y) = l_\searrow(x)l_\nearrow(y),$$

$$(4.7) \quad r_\swarrow(y)r_\prec(x) = r_\swarrow(x \vee y), r_\swarrow(y)l_\prec(x) = l_\swarrow(x)r_\vee(y), l_\swarrow(x \prec y) = l_\swarrow(x)l_\vee(y),$$

$$(4.8) \quad r_\swarrow(y)r_\succ(x) = r_\searrow(x \swarrow y), r_\swarrow(y)l_\succ(x) = l_\searrow(x)r_\swarrow(y), l_\swarrow(x \succ y) = l_\searrow(x)l_\swarrow(y),$$

$$(4.9) \quad r_\searrow(y)r_\star(x) = r_\searrow(x \searrow y), r_\searrow(y)l_\star(x) = l_\searrow(x)r_\searrow(y), l_\searrow(x \star y) = l_\searrow(x)l_\star(y).$$

In fact, $(V, l_\nwarrow, r_\nwarrow, l_\nearrow, r_\nearrow, l_\swarrow, r_\swarrow, l_\searrow, r_\searrow)$ is a bimodule of a quadri-algebra A if and only if the direct sum $A \oplus V$ of the underlying vector spaces of A and V is turned into a quadri-algebra (the *semidirect sum*) by defining multiplications in $A \oplus V$ by (we still denote them by $\nwarrow, \nearrow, \swarrow, \searrow$):

$$(4.10) \quad (x_1 + u_1) \circ (x_2 + u_2) := x_1 \circ x_2 + (l_\circ(x_1)u_2 + r_\circ(x_2)u_1), \forall x_1, x_2 \in A, u_1, u_2 \in V, \circ \in \{\nwarrow, \nearrow, \swarrow, \searrow\}.$$

We denote it by $A \ltimes_{l_\nwarrow, r_\nwarrow, l_\nearrow, r_\nearrow, l_\swarrow, r_\swarrow, l_\searrow, r_\searrow} V$ or simply $A \ltimes V$.

Lemma 4.2. ([Bai3]) Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra. If $(V, l_\nwarrow, r_\nwarrow, l_\nearrow, r_\nearrow, l_\swarrow, r_\swarrow, l_\searrow, r_\searrow)$ is a bimodule of A , then $(V^*, r_{\nwarrow}^*, l_{\nwarrow}^*, -r_{\vee}^*, -l_{\vee}^*, -r_{\succ}^*, -l_{\succ}^*, r_{\wedge}^*, l_{\wedge}^*)$ is a bimodule of A .

$$(4.11) \quad (x+a) \diamond (y+b) := x \diamond_A y + l_{\diamond_B}(a)y + r_{\diamond_B}(b)x + a \diamond_B b + l_{\diamond_A}(x)b + r_{\diamond_A}(y)a, \diamond \in \{\nwarrow, \nearrow, \swarrow, \searrow\},$$

define a quadri-algebra structure on $A \oplus B$. Then $(A, B, l_{\nwarrow A}, r_{\nwarrow A}, l_{\nearrow A}, r_{\nearrow A}, l_{\swarrow A}, r_{\swarrow A}, l_{\searrow A}, r_{\searrow A}, l_{\nwarrow B}, r_{\nwarrow B}, l_{\nearrow B}, r_{\nearrow B}, l_{\swarrow B}, r_{\swarrow B}, l_{\searrow B}, r_{\searrow B})$ is called a *matched pair of quadri-algebras* and the quadri-algebra structure on $A \oplus B$ is denoted by $A \bowtie_{l_{\nwarrow A}, r_{\nwarrow A}, l_{\nearrow A}, r_{\nearrow A}, l_{\swarrow A}, r_{\swarrow A}, l_{\searrow A}, r_{\searrow A}}^{l_{\nwarrow B}, r_{\nwarrow B}, l_{\nearrow B}, r_{\nearrow B}, l_{\swarrow B}, r_{\swarrow B}, l_{\searrow B}, r_{\searrow B}} B$ or simply $A \bowtie B$.

Remark 4.4. Similar to Proposition 3.4 one can also write down the necessary and sufficient conditions that the above linear maps make $A \oplus B$ into a quadri-algebra. We omit the details since we will not use such a conclusion in this paper directly. In particular, in this case, A and B are bimodules of B and A respectively.

Proposition 4.5. *Let $(A, B, l_{\nwarrow A}, r_{\nwarrow A}, l_{\nearrow A}, r_{\nearrow A}, l_{\swarrow A}, r_{\swarrow A}, l_{\searrow A}, r_{\searrow A}, l_{\nwarrow B}, r_{\nwarrow B}, l_{\nearrow B}, r_{\nearrow B}, l_{\swarrow B}, r_{\swarrow B}, l_{\searrow B}, r_{\searrow B})$ be a matched pair of quadri-algebras. Then $((A)_v, (B)_v, l_{\nwarrow A} + l_{\nearrow A}, r_{\nwarrow A} + r_{\nearrow A}, l_{\swarrow A} + l_{\searrow A}, r_{\swarrow A} + r_{\searrow A}, l_{\nwarrow B} + l_{\nearrow B}, r_{\nwarrow B} + r_{\nearrow B}, l_{\swarrow B} + l_{\searrow B}, r_{\swarrow B} + r_{\searrow B})$ is a matched pair of dendriform dialgebras.*

Proof. It is straightforward. \square

Proposition 4.6. *Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra. Suppose that there is a quadri-algebra structure $(\nwarrow_*, \nearrow_*, \swarrow_*, \searrow_*)$ on the dual space A^* . Then $(A_v, (A^*)_v, -R_{\swarrow}^*, L_{\swarrow}^*, R_{\nwarrow}^*, -L_{\nearrow}^*, -R_{\swarrow_*}^*, L_{\swarrow_*}^*, R_{\nwarrow_*}^*, -L_{\nearrow_*}^*)$ is a matched pair of dendriform dialgebras if and only if $(A, A^*, R_{\nwarrow}^*, L_{\star}^*, -R_{\vee}^*, -L_{\swarrow}^*, -R_{\swarrow}^*, -L_{\wedge}^*, R_{\star}^*, L_{\nwarrow}^*, R_{\nwarrow_*}^*, L_{\star_*}^*, -R_{\vee_*}^*, -L_{\swarrow_*}^*, -R_{\swarrow_*}^*, -L_{\wedge_*}^*, R_{\star_*}^*, L_{\nwarrow_*}^*)$ is a matched pair of quadri-algebras.*

Proof. By Proposition 4.5, we only need to prove the “only if” part of the proposition. In fact, if $(A_v, (A^*)_v, -R_{\swarrow}^*, L_{\searrow}^*, R_{\swarrow}^*, -L_{\nearrow}^*, -R_{\swarrow*}^*, L_{\searrow*}^*, R_{\swarrow*}^*, -L_{\nearrow*}^*)$ is a matched pair of dendriform dialgebras, then from Proposition 3.5, we show that $(A_v \oplus (A^*)_v, A_v, (A^*)_v, \mathfrak{B}_S)$ is a Manin triple of dendriform dialgebras. Hence there exists a quadri-algebra structure $(\lhd_{\bullet}, \rhd_{\bullet}, \lhd_{\bullet}, \rhd_{\bullet})$ on $A_v \bowtie_{\substack{-R_{\swarrow*}^*, L_{\searrow*}^*, R_{\swarrow*}^*, -L_{\nearrow*}^* \\ -R_{\swarrow}^*, L_{\searrow}^*, R_{\swarrow}^*, -L_{\nearrow}^*}} (A^*)_v$ which is given by Proposition 2.11. Moreover, A and A^* are the sub-quadri-algebras. For any $x, y \in A, a^*, b^* \in A^*$, we have

$$\langle x \curvearrowright_{\bullet} a^*, y \rangle = \mathfrak{B}_S(x, a^* \star_{\bullet} y) = \langle x, L^*_{\curvearrowright}(y) a^* \rangle = \langle R^*_{\curvearrowright}(x) a^*, y \rangle,$$

$$\langle x \curvearrowright_{\bullet} a^*, b^* \rangle = \mathfrak{B}_S(x, a^* \star_* b^*) = \langle L_{\star_*}^*(a^*)x, b^* \rangle.$$

So $x \curvearrowright_{\bullet} a^* = R_{\searrow}^*(x)a^* + L_{\star*}^*(a^*)x$. Similarly, we show that

$$x \nearrow_{\bullet} a^* = -R_{\searrow}^*(x)a^* - L_{\searrow}^*(a^*)x, x \swarrow_{\bullet} a^* = -R_{\swarrow}^*(x)a^* - L_{\swarrow}^*(a^*)x,$$

$$\begin{aligned} x \searrow_{\bullet} a^* &= R_{\star}^*(x)a^* + L_{\searrow_{\star}}^*(a^*)x, a^* \nwarrow_{\bullet} x = R_{\nwarrow_{\star}}^*(a^*)x + L_{\star}^*(x)a^*, \\ a^* \nearrow_{\bullet} x &= -R_{\vee_{\star}}^*(a^*)x - L_{\nearrow_{\star}}^*(x)a^*, a^* \swarrow_{\bullet} x = -R_{\swarrow_{\star}}^*(a^*)x - L_{\wedge_{\star}}^*(x)a^*, \end{aligned}$$

and $a^* \searrow_{\bullet} x = R_{\star}^*(a^*)x + L_{\searrow_{\star}}^*(x)a^*$. Therefore, $(A, A^*, R_{\searrow_{\star}}^*, L_{\star}^*, -R_{\vee_{\star}}^*, -L_{\nearrow_{\star}}^*, -R_{\swarrow_{\star}}^*, -L_{\wedge_{\star}}^*, R_{\star}^*, L_{\searrow_{\star}}^*, R_{\nwarrow_{\star}}^*, L_{\star}^*, -R_{\vee_{\star}}^*, -L_{\nearrow_{\star}}^*, -R_{\swarrow_{\star}}^*, -L_{\wedge_{\star}}^*, R_{\star}^*, L_{\searrow_{\star}}^*)$ is a matched pair of quadri-algebras. \square

In fact, the above equivalence between two matched pairs can be interpreted in terms of their corresponding Manin triples as follows.

Definition 4.7. Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra and let \mathfrak{B} be a symmetric bilinear form. If \mathfrak{B} satisfies Eqs. (2.14)-(2.15), then \mathfrak{B} is called *invariant* on A .

Proposition 4.8. ([Bai3]) *Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra and let \mathfrak{B} be a symmetric bilinear form. If \mathfrak{B} is invariant on A , then \mathfrak{B} is a 2-cocycle of the associated vertical dendriform dialgebra (A_v, \wedge, \vee) . Conversely, if \mathfrak{B} is a nondegenerate 2-cocycle of a dendriform dialgebra, then \mathfrak{B} is invariant on the quadri-algebra given by Eqs. (2.14)-(2.15).*

Definition 4.9. Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra. If there is a quadri-algebra structure on the direct sum of the underlying vector space of A and A^* such that A and A^* are quadri-subalgebras and the bilinear form \mathfrak{B}_S on $A \oplus A^*$ given by Eq. (3.2) is invariant, then $(A \bowtie A^*, A, A^*, \mathfrak{B}_S)$ is called a (standard) *Manin triple of quadri-algebras associated to a nondegenerate invariant bilinear form*.

By Proposition 4.8, the following conclusion is obvious:

Corollary 4.10. *$(A \bowtie A^*, A, A^*, \mathfrak{B}_S)$ is a Manin triple of quadri-algebras associated to a nondegenerate invariant bilinear form if and only if $(A_v \bowtie (A_v)^*, A_v, A_v^*, \mathfrak{B}_S)$ is Manin triple of dendriform dialgebras associated to a nondegenerate 2-cocycle.*

Remark 4.11. By Proposition 4.6, it is obvious that a Manin triple of quadri-algebras associated to a nondegenerate invariant bilinear form can be interpreted in terms of a matched pair of quadri-algebras (cf. Theorem 5.4).

5. QUADRI-BIALGEBRAS

Proposition 5.1. *Let $(A, \nwarrow, \nearrow, \swarrow, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ be a quadri-algebra equipped with four cooperations $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} : A \rightarrow A \otimes A$. Suppose that $\alpha^*, \beta^*, \tilde{\alpha}^*, \tilde{\beta}^* : A^* \otimes A^* \subset (A \otimes A)^* \rightarrow A^*$ induce a quadri-algebra structure on A^* . Set $\nwarrow_{\star} := \alpha^*, \nearrow_{\star} := \beta^*, \swarrow_{\star} := \tilde{\alpha}^*, \searrow_{\star} := \tilde{\beta}^*$. Then $(A_v, (A^*)_v, -R_{\nwarrow_{\star}}^*, L_{\nwarrow_{\star}}^*, R_{\nwarrow_{\star}}^*, -L_{\nearrow_{\star}}^*, -R_{\swarrow_{\star}}^*, L_{\swarrow_{\star}}^*, R_{\swarrow_{\star}}^*, -L_{\searrow_{\star}}^*)$ is a matched pair of dendriform dialgebras if and only if the following equations hold:*

$$(5.1) \quad \tilde{\alpha}(x \star y) = (R_{\wedge}(y) \otimes 1)\tilde{\alpha}(x) + (1 \otimes L_{\vee}(x))\tilde{\alpha}(y),$$

$$(5.2) \quad \beta(x \star y) = (R_{\nearrow}(y) \otimes 1)\beta(x) + (1 \otimes L_{\vee}(x))\beta(y),$$

$$(5.3) \quad (\alpha + \tilde{\alpha})(x \wedge y) = (R_{\wedge}(y) \otimes 1)(\alpha + \tilde{\alpha})(x) + (1 \otimes L_{\nearrow}(x))\tilde{\alpha}(y),$$

$$(5.4) \quad (\beta + \tilde{\beta})(x \wedge y) = (R_{\nwarrow}(y) \otimes 1)(\beta + \tilde{\beta})(x) + (1 \otimes L_{\wedge}(x))\tilde{\beta}(y),$$

$$(5.5) \quad (\beta + \tilde{\beta})(x \vee y) = (1 \otimes L_{\vee}(x))(\beta + \tilde{\beta})(y) + (R_{\swarrow}(y) \otimes 1)\beta(x),$$

$$(5.6) \quad (\alpha + \tilde{\alpha})(x \vee y) = (1 \otimes L_{\searrow}(x))(\alpha + \tilde{\alpha})(y) + (R_{\vee}(y) \otimes 1)\alpha(x),$$

$$(5.7) \quad (\alpha + \beta)(x \succ y) = (1 \otimes L_{\searrow}(x))(\alpha + \beta)(y) + (R_{\succ}(y) \otimes 1)\alpha(x),$$

$$(5.8) \quad (\tilde{\alpha} + \tilde{\beta})(x \succ y) = (1 \otimes L_{\succ}(x))(\tilde{\alpha} + \tilde{\beta})(y) + (R_{\nearrow}(y) \otimes 1)\tilde{\alpha}(x),$$

$$(5.9) \quad (\alpha + \beta)(x \prec y) = (R_{\prec}(y) \otimes 1)(\alpha + \beta)(x) + (1 \otimes L_{\swarrow}(x))\beta(y),$$

$$(5.10) \quad (\tilde{\alpha} + \tilde{\beta})(x \prec y) = (R_{\nwarrow}(y) \otimes 1)(\tilde{\alpha} + \tilde{\beta})(x) + (1 \otimes L_{\prec}(x))\tilde{\beta}(y),$$

$$(5.11) \quad (1 \otimes L_{\succ}(y) - R_{\wedge}(y) \otimes 1)\tau\beta(x) = (1 \otimes R_{\prec}(x) - L_{\vee}(x) \otimes 1)\tilde{\alpha}(y),$$

$$(5.12) \quad (1 \otimes R_{\nwarrow}(x) - L_{\vee}(x) \otimes 1)(\alpha + \tilde{\alpha})(y) = (1 \otimes L_{\nearrow}(y))\tau\beta(x) - (R_{\vee}(y) \otimes 1)\tau\tilde{\beta}(x),$$

$$(5.13) \quad (R_{\wedge}(y) \otimes 1 - 1 \otimes L_{\searrow}(y))(\tau\beta + \tau\tilde{\beta})(x) = (L_{\wedge}(x) \otimes 1)\alpha(y) - (1 \otimes R_{\swarrow}(x))\tilde{\alpha}(y),$$

$$(5.14) \quad (1 \otimes L_{\succ}(x) - R_{\nwarrow}(x) \otimes 1)(\tau\alpha + \tau\beta)(y) = (1 \otimes R_{\succ}(y))\tilde{\beta}(x) - (L_{\swarrow}(y) \otimes 1)\tilde{\alpha}(x),$$

$$(5.15) \quad (1 \otimes L_{\searrow}(x) - R_{\prec}(x) \otimes 1)(\tau\tilde{\alpha} + \tau\tilde{\beta})(y) = (1 \otimes R_{\nearrow}(y))\beta(x) - (L_{\prec}(y) \otimes 1)\alpha(x),$$

$$(5.16) \quad (\alpha + \tilde{\alpha} + \beta + \tilde{\beta})(x \swarrow y) = (R_{\swarrow}(y) \otimes 1)(\alpha + \beta)(x) + (1 \otimes L_{\swarrow}(x))(\beta + \tilde{\beta})(y),$$

$$(5.17) \quad (\alpha + \beta + \tilde{\alpha} + \tilde{\beta})(x \nearrow y) = (R_{\nearrow}(y) \otimes 1)(\alpha + \tilde{\alpha})(x) + (1 \otimes L_{\nearrow}(x))(\tilde{\alpha} + \tilde{\beta})(y).$$

$$(5.18) \quad (L_{\swarrow}(y) \otimes 1)(\alpha + \tilde{\alpha})(x) + (1 \otimes L_{\nearrow}(x))(\tau\alpha + \tau\beta)(y) = (R_{\swarrow}(x) \otimes 1)(\tau\tilde{\alpha} + \tau\tilde{\beta})(y) + (1 \otimes R_{\nearrow}(y))(\beta + \tilde{\beta})(x).$$

Proof. From Proposition 3.5, we need to prove Eqs. (5.1)-(5.18) are equivalent to Eqs. (3.3)-(3.20) respectively in the case that $A = A_v, B = (A^*)_v$ and $l_{\wedge_A} = -R_{\swarrow}^*, r_{\wedge_A} = L_{\succ}^*, l_{\vee_A} = R_{\prec}^*, r_{\vee_A} = -L_{\nearrow}^*, l_{\wedge_B} = -R_{\swarrow}^*, r_{\wedge_B} = L_{\succ}^*, l_{\vee_B} = R_{\prec}^*, r_{\vee_B} = -L_{\nearrow}^*$. As an example, we give an explicit proof that

$$(5.19) \quad L_{\succ}^*(a^*)(x \wedge y) - L_{\succ}^*(R_{\nwarrow}^*(y)a^*)x - x \wedge (L_{\searrow}^*(a^*)y) = 0,$$

holds if and only if Eq. (5.4) holds. The other equivalences are similar. In fact, let the left hand side of Eq. (5.19) act on an arbitrary element $b^* \in A^*$. Then we have

$$\begin{aligned} & \langle L_{\succ}^*(a^*)(x \wedge y) - L_{\succ}^*(R_{\nwarrow}^*(y)a^*)x - x \wedge (L_{\searrow}^*(a^*)y), b^* \rangle \\ &= \langle x \wedge y, a^* \succ_* b^* \rangle - \langle x, (R_{\nwarrow}^*(y)a^*) \succ_* b^* \rangle - \langle y, a^* \searrow_* (L_{\wedge}^*(x)b^*) \rangle \\ &= \langle (\beta + \tilde{\beta})(x \wedge y) - (R_{\nwarrow}(y) \otimes 1)(\beta + \tilde{\beta})(x) - (1 \otimes L_{\wedge}(x))\tilde{\beta}(y), a^* \otimes b^* \rangle. \end{aligned}$$

So the conclusion follows. \square

Definition 5.2. (1) Let $(A, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ be a vector space with four comultiplications $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} : A \rightarrow A \otimes A$. If $(A^*, \alpha^*, \beta^*, \tilde{\alpha}^*, \tilde{\beta}^*)$ becomes a quadri-algebra, then we call $(A, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ a *quadri-coalgebra*, where $\alpha^*, \beta^*, \tilde{\alpha}^*, \tilde{\beta}^* : A^* \otimes A^* \subset (A \otimes A)^* \rightarrow A^*$. A *homomorphism* between two quadri-coalgebras is defined as a linear map (between the two quadri-coalgebras) which preserves the cooperations, respectively.

(2) Let $(A, \lrcorner, \nearrow, \swarrow, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ be a quadri-algebra equipped with four cooperations $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} : A \rightarrow A \otimes A$ such that $(A, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ is a quadri-coalgebra and $\alpha, \beta, \tilde{\alpha}$ and $\tilde{\beta}$ satisfy Eqs. (5.1)-(5.18), then $(A, \lrcorner, \nearrow, \swarrow, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ is called a *quadri-bialgebra*. A *homomorphism* between two quadri-bialgebras is defined as a linear map (between the two quadri-bialgebras) which is a homomorphisms of both quadri-algebras and quadri-coalgebras.

Combining Proposition 5.1 and the discussion in the previous sections, we have the following conclusion:

Theorem 5.3. *Let $(A, \lrcorner_A, \nearrow_A, \swarrow_A, \searrow_A, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ be a quadri-algebra with four comultiplications $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} : A \rightarrow A \otimes A$, such that $(A, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ is a quadri-coalgebra. Then the following conditions are equivalent ($\lrcorner_B := \alpha^*, \nearrow_B := \beta^*, \swarrow_B := \tilde{\alpha}^*, \searrow_B := \tilde{\beta}^*$, and \mathfrak{B}_S is given by Eq. (3.2)):*

- (1) $(A_v \bowtie (A^*)_v, A_v, (A^*)_v, \mathfrak{B}_S)$ is a Manin triple of dendriform dialgebras associated to a nondegenerate 2-cocycle;
- (2) $(A \bowtie A^*, A, A^*, \mathfrak{B}_S)$ is a Manin triple of quadri-algebras associated to a nondegenerate invariant bilinear form.
- (3) $(A_v, (A^*)_v, -R_{\swarrow_A}^*, L_{\nearrow_A}^*, R_{\lrcorner_A}^*, -L_{\nearrow_A}^*, -R_{\swarrow_B}^*, L_{\nearrow_B}^*, R_{\lrcorner_B}^*, -L_{\nearrow_B}^*)$ is a matched pair of dendriform dialgebras;
- (4) $(A, A^*, R_{\searrow_A}^*, L_{\star_A}^*, -R_{\vee_A}^*, -L_{\lrcorner_A}^*, -R_{\nearrow_A}^*, -L_{\wedge_A}^*, R_{\star_A}^*, L_{\lrcorner_A}^*, R_{\searrow_B}^*, L_{\star_B}^*, -R_{\vee_B}^*, -L_{\lrcorner_B}^*, -R_{\nearrow_B}^*, -L_{\wedge_B}^*, R_{\star_B}^*, L_{\lrcorner_B}^*)$ is a matched pair of quadri-algebras;
- (5) $(A, \lrcorner_A, \nearrow_A, \swarrow_A, \searrow_A, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ is a quadri-bialgebra.

By a standard proof (cf. [Bai2], Proposition 2.2.10), we get the following result:

Proposition 5.4. *Two Manin triples of dendriform dialgebras are isomorphic if and only if their corresponding quadri-bialgebras are isomorphic.*

Remark 5.5. It is obvious that for a quadri-bialgebra $(A, \lrcorner, \nearrow, \swarrow, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$, the dual $(A^*, \alpha^*, \beta^*, \tilde{\alpha}^*, \tilde{\beta}^*, \gamma, \delta, \tilde{\gamma}, \tilde{\delta})$ is also a quadri-bialgebra, where $\gamma^* = \lrcorner, \delta^* = \nearrow, \tilde{\gamma}^* = \swarrow, \tilde{\delta}^* = \searrow$.

6. COBOUNDARY QUADRI-BIALGEBRAS

Definition 6.1. A quadri-bialgebra $(A, \lrcorner, \rhd, \lhd, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ is called *coboundary* if $\alpha, \beta, \tilde{\alpha}$ and $\tilde{\beta}$ satisfy

$$(6.1) \quad \alpha(x) = (-1 \otimes L_{\searrow}(x) + R_{\star}(x) \otimes 1)r_{\lrcorner},$$

$$(6.2) \quad \beta(x) = (1 \otimes L_{\vee}(x) - R_{\lhd}(x) \otimes 1)r_{\rhd},$$

$$(6.3) \quad \tilde{\alpha}(x) = (1 \otimes L_{\succ}(x) - R_{\wedge}(x) \otimes 1)r_{\lhd},$$

$$(6.4) \quad \tilde{\beta}(x) = (-1 \otimes L_{\star}(x) + R_{\lrcorner}(x) \otimes 1)r_{\searrow},$$

where $r_{\lrcorner}, r_{\rhd}, r_{\lhd}, r_{\searrow} \in A \otimes A$ and $x \in A$.

Proposition 6.2. Let $(A, \lrcorner, \rhd, \lhd, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ be a quadri-algebra with four comultiplications $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ defined by Eqs. (6.1)-(6.4) respectively. If $r_{\lrcorner} = r_{\rhd} = r_{\lhd} = r_{\searrow} = r \in A \otimes A$ and r is skew-symmetric, then $\alpha, \beta, \tilde{\alpha}$ and $\tilde{\beta}$ satisfy Eqs. (5.1)-(5.18).

Proof. Straightforward. □

Lemma 6.3. Let A be a vector space and $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} : A \rightarrow A \otimes A$ be four cooperations. Then $(A, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ is a quadri-coalgebra if and only if the linear maps $R_i : A \rightarrow A \otimes A \otimes A$ ($i \in \{1, \dots, 9\}$) defined by the following equations are zero:

$$(6.5) \quad R_1(x) := (\alpha \otimes 1)\alpha(x) - (1 \otimes (\alpha + \beta + \tilde{\alpha} + \tilde{\beta}))\alpha(x),$$

$$(6.6) \quad R_2(x) := (\beta \otimes 1)\alpha(x) - (1 \otimes (\alpha + \tilde{\alpha}))\beta(x),$$

$$(6.7) \quad R_3(x) := ((\alpha + \beta) \otimes 1)\beta(x) - (1 \otimes (\beta + \tilde{\beta}))\beta(x),$$

$$(6.8) \quad R_4(x) := (\tilde{\alpha} \otimes 1)\alpha(x) - (1 \otimes (\alpha + \beta))\tilde{\alpha}(x),$$

$$(6.9) \quad R_5(x) := (\tilde{\beta} \otimes 1)\alpha(x) - (1 \otimes \alpha)\tilde{\beta}(x),$$

$$(6.10) \quad R_6(x) := ((\tilde{\alpha} + \tilde{\beta}) \otimes 1)\beta(x) - (1 \otimes \beta)\tilde{\beta}(x),$$

$$(6.11) \quad R_7(x) := ((\alpha + \tilde{\alpha}) \otimes 1)\tilde{\alpha}(x) - (1 \otimes (\tilde{\alpha} + \tilde{\beta}))\tilde{\alpha}(x),$$

$$(6.12) \quad R_8(x) := ((\beta + \tilde{\beta}) \otimes 1)\tilde{\alpha}(x) - (1 \otimes \tilde{\alpha})\tilde{\beta}(x),$$

$$(6.13) \quad R_9(x) := ((\alpha + \beta + \tilde{\alpha} + \tilde{\beta}) \otimes 1)\tilde{\beta}(x) - (1 \otimes \tilde{\beta})\tilde{\beta}(x).$$

Proof. It follows from the definition of a quadri-algebra. □

Definition 6.4. Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra and let $r \in A \otimes A$. The following equations are called Q_i^j -equations ($i = 1, 2, 3; j = 1, 2$) respectively:

$$(6.14) \quad Q_1^1 := r_{23} \wedge r_{12} - r_{13} \succ r_{23} + r_{12} \swarrow r_{13} = 0,$$

$$(6.15) \quad Q_1^2 := r_{23} \vee r_{12} - r_{12} \prec r_{13} + r_{13} \nearrow r_{23} = 0,$$

$$(6.16) \quad Q_2^1 := r_{12} \wedge r_{13} - r_{23} \succ r_{12} - r_{13} \swarrow r_{23} = 0,$$

$$(6.17) \quad Q_2^2 := r_{12} \vee r_{13} + r_{13} \prec r_{23} + r_{23} \nearrow r_{12} = 0,$$

$$(6.18) \quad Q_3^1 := r_{13} \wedge r_{23} + r_{12} \succ r_{13} + r_{23} \swarrow r_{12} = 0,$$

$$(6.19) \quad Q_3^2 := r_{13} \vee r_{23} - r_{23} \prec r_{12} - r_{12} \nearrow r_{13} = 0.$$

Proposition 6.5. Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra and let $r \in A \otimes A$. Define $\alpha, \beta, \tilde{\alpha}$ and $\tilde{\beta}$ by Eqs. (6.1)-(6.4) respectively, where $r_{\nwarrow} = r_{\nearrow} = r_{\swarrow} = r_{\searrow} = r$. Then $(A, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ becomes a quadri-coalgebra if and only if the following equations holds (for any $x \in A$):

$$(6.20) \quad (1 \otimes 1 \otimes L_{\searrow}(x) - R_{\star}(x) \otimes 1 \otimes 1)(Q_1^2 - Q_3^1) = 0,$$

$$(6.21) \quad (1 \otimes 1 \otimes L_{\swarrow}(x) - R_{\prec}(x) \otimes 1 \otimes 1)Q_1^2 = 0,$$

$$(6.22) \quad (1 \otimes 1 \otimes L_{\vee}(x) - R_{\swarrow}(x) \otimes 1 \otimes 1)Q_3^1 = 0,$$

$$(6.23) \quad (1 \otimes 1 \otimes L_{\nwarrow}(x) - R_{\wedge}(x) \otimes 1 \otimes 1)Q_2^1 = 0,$$

$$(6.24) \quad (1 \otimes 1 \otimes L_{\swarrow}(x) - R_{\nwarrow}(x) \otimes 1 \otimes 1)(Q_2^1 + Q_3^2) = 0,$$

$$(6.25) \quad (1 \otimes 1 \otimes L_{\vee}(x) - R_{\nwarrow}(x) \otimes 1 \otimes 1)Q_3^2 = 0,$$

$$(6.26) \quad (1 \otimes 1 \otimes L_{\succ}(x) - R_{\wedge}(x) \otimes 1 \otimes 1)Q_2^2 = 0,$$

$$(6.27) \quad (1 \otimes 1 \otimes L_{\succ}(x) - R_{\nwarrow}(x) \otimes 1 \otimes 1)Q_1^1 = 0,$$

$$(6.28) \quad (1 \otimes 1 \otimes L_{\star}(x) - R_{\nwarrow}(x) \otimes 1 \otimes 1)(Q_3^1 + Q_3^2) = 0.$$

Proof. In fact, we need to prove $R_i(x) = 0, 1 \leq i \leq 9$, if and only if Eqs. (6.20)-(6.28) hold respectively. As an example we give an explicit proof that $R_2(x) = 0$ if and only if Eq. (6.21)

holds. The proof of the other cases is similar. In fact, after rearranging the terms suitably, we divide $R_2(x)$ into three parts: $R_2(x) = (R1) + (R2) + (R3)$, where

$$\begin{aligned}
(R1) &= \sum_{i,j} \{-a_i \otimes a_j \vee b_i \otimes x \searrow b_j + a_i \prec a_j \otimes b_i \otimes x \searrow b_j + a_j \otimes a_i \otimes (x \vee b_j) \searrow b_i \\
&\quad - a_j \otimes a_i \otimes (x \vee b_j) \succ b_i\} \\
&= (1 \otimes 1 \otimes L_{\searrow}(x))(-r_{23} \vee r_{12} + r_{12} \prec r_{13} - r_{13} \nearrow r_{23}), \\
(R2) &= \sum_{i,j} \{a_i \otimes (a_j \star x) \vee b_i \otimes b_j - a_j \otimes a_i \star (x \vee b_j) \otimes b_i + a_j \otimes a_i \wedge (x \vee b_j) \otimes b_i\} = 0, \\
(R3) &= \sum_{i,j} \{-a_i \prec (a_j \star x) \otimes b_i \otimes b_j - a_j \prec x \otimes a_i \otimes b_j \searrow b_i + a_j \prec x \otimes a_i \star b_j \otimes b_i \\
&\quad + a_j \prec x \otimes a_i \otimes b_j \succ b_i - a_j \prec x \otimes a_i \wedge b_j \otimes b_i \\
&= (R_{\prec}(x) \otimes 1 \otimes 1)(-r_{12} \prec r_{13} + r_{13} \nearrow r_{23} + r_{23} \vee r_{12}).
\end{aligned}$$

So the conclusion follows. \square

By the above study, we have the following conclusion.

Theorem 6.6. *Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra and let $r \in A \otimes A$ be skew-symmetric. Then the comultiplications $\alpha, \beta, \tilde{\alpha}$ and $\tilde{\beta}$ defined by Eqs.(6.1)-(6.4) with $r_{\nwarrow} = r_{\nearrow} = r_{\swarrow} = r_{\searrow} = r$ respectively make $(A, \nwarrow, \nearrow, \swarrow, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ into a quadri-bialgebra if and only if Eqs. (6.20)-(6.28) are satisfied.*

Moreover, we have the following ‘‘Drinfeld’s double’’ construction of a quadri-bialgebra ([CP]).

Theorem 6.7. *Let $(A, \nwarrow, \nearrow, \swarrow, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ be a quadri-bialgebra. Then there exists a canonical quadri-bialgebra structure on $A \oplus A^*$ such that the inclusion $i_1 : A \rightarrow A \oplus A^*$ is a homomorphism of quadri-bialgebras, which is from $(A, \nwarrow, \nearrow, \swarrow, \searrow, -\alpha, -\beta, -\tilde{\alpha}, -\tilde{\beta})$ to $A \oplus A^*$, and the inclusion $i_2 : A^* \rightarrow A \oplus A^*$ is also a homomorphism of quadri-bialgebras, which is from the quadri-bialgebra given in Remark 5.5 to $A \oplus A^*$.*

Proof. Let $r = \sum_i e_i \otimes e_i^*$ correspond to the identity map $\text{id} : A \rightarrow A$, where $\{e_i, \dots, e_s\}$ is a basis of A and $\{e_1^*, \dots, e_s^*\}$ is the dual basis. Suppose the quadri-algebra structure $(\nwarrow_{\bullet}, \nearrow_{\bullet}, \swarrow_{\bullet}, \searrow_{\bullet})$ on $A \oplus A^*$ is given by $\mathcal{QD}(A) := A \bowtie_{R_{\nwarrow}^*, L_{\nwarrow}^*, -R_{\vee}^*, -L_{\prec}^*, -R_{\succ}^*, -L_{\wedge}^*, R_{\star}^*, L_{\nwarrow}^*} A^*$, where the subscript $*$ is used to denote the quadri-algebra structure on A^* . Then for any $x, y \in A, a^*, b^* \in A^*$,

$$\begin{aligned}
x \nwarrow_{\bullet} y &= x \nwarrow y, x \nearrow_{\bullet} y = x \nearrow y, x \swarrow_{\bullet} y = x \swarrow y, x \searrow_{\bullet} y = x \searrow y, \\
a^* \nwarrow_{\bullet} b^* &= a^* \nwarrow_{\star} b^*, a^* \nearrow_{\bullet} b^* = a^* \nearrow_{\star} b^*, a^* \swarrow_{\bullet} b^* = a^* \swarrow_{\star} b^*, a^* \searrow_{\bullet} b^* = a^* \searrow_{\star} b^*, \\
x \nwarrow_{\bullet} a^* &= R_{\nwarrow}^*(x)a^* + L_{\star}^*(a^*)x, x \nearrow_{\bullet} a^* = -R_{\vee}^*(x)a^* - L_{\prec}^*(a^*)x, x \swarrow_{\bullet} a^* = -R_{\succ}^*(x)a^* - L_{\wedge}^*(a^*)x, \\
x \searrow_{\bullet} a^* &= R_{\star}^*(x)a^* + L_{\nwarrow}^*(a^*)x, a^* \nwarrow_{\bullet} x = R_{\nwarrow}^*(a^*)x + L_{\star}^*(x)a^*, a^* \nearrow_{\bullet} x = -R_{\vee}^*(a^*)x - L_{\prec}^*(x)a^*, \\
a^* \swarrow_{\bullet} x &= -R_{\succ}^*(a^*)x - L_{\wedge}^*(x)a^*, a^* \searrow_{\bullet} x = R_{\star}^*(a^*)x + L_{\nwarrow}^*(x)a^*.
\end{aligned}$$

We prove that r satisfies Eqs. (5.1)-(5.18). We give an explicit proof that r satisfy Eq. (5.13) as an example. The proof of the other cases is similar.

For any $\mu, \nu \in \mathcal{QD}(A)$, Eq. (5.13) is equivalent to

$$\begin{aligned} \sum_k \{ & -(\mu \wedge_\bullet e_k^*) \wedge_\bullet \nu \otimes e_k - e_k^* \wedge_\bullet \nu \otimes e_k \swarrow_\bullet \mu + \mu \wedge_\bullet e_k^* \otimes \nu \searrow_\bullet e_k + e_k^* \otimes \nu \searrow_\bullet (e_k \swarrow_\bullet \mu) \\ & + \mu \wedge e_k \otimes \nu \searrow_\bullet e_k^* - \mu \wedge_\bullet (e_k \star_\bullet \nu) \otimes e_k^* + e_k \otimes (\nu \succ_\bullet e_k^*) \swarrow_\bullet \mu - e_k \wedge_\bullet \nu \otimes e_k^* \swarrow_\bullet \mu \} = 0. \end{aligned}$$

We can prove the equation in the following cases: (I) $\mu, \nu \in A$; (II) $\mu, \nu \in A^*$; (III) $\mu \in A, \nu \in A^*$; (IV) $\mu \in A^*, \nu \in A$. As an example, we give the proof of the last case. The proof of the other cases is similar. Let $\mu = e_i^*, \nu = e_j$, then for any m, n , the coefficient of $e_m^* \otimes e_n$ is

$$\begin{aligned} & \sum_k \langle -(e_i^* \wedge_\bullet e_n^*) \wedge_\bullet e_j, e_m \rangle - \langle e_k^* \wedge_\bullet e_j, e_m \rangle \langle e_k \swarrow_\bullet e_i^*, e_n^* \rangle + \langle e_i^* \wedge_\bullet e_k^*, e_m \rangle \langle e_j \searrow_\bullet e_k, e_n^* \rangle \\ & + \langle e_j \searrow_\bullet (e_m \swarrow_\bullet e_i^*), e_n^* \rangle + \langle e_i^* \wedge_\bullet e_k, e_m \rangle \langle e_j \searrow_\bullet e_k^*, e_n^* \rangle \\ = & \sum_k -\langle e_i^* \wedge_\bullet e_n^*, e_j \succ e_m \rangle + \langle e_k^*, e_j \succ e_m \rangle \langle e_k, e_i^* \wedge_\bullet e_n^* \rangle + \langle e_i^* \wedge_\bullet e_k^*, e_m \rangle \langle e_j \searrow e_k, e_n^* \rangle \\ & - \langle e_i^*, e_k \succ e_m \rangle \langle e_j, e_k^* \nwarrow_\bullet e_n^* \rangle - \langle e_m, e_i^* \wedge_\bullet e_k^* \rangle \langle e_j \searrow e_k, e_n^* \rangle + \langle e_i^*, e_k \succ e_m \rangle \langle e_j, e_k^* \nwarrow_\bullet e_n^* \rangle \\ = & 0. \end{aligned}$$

Similarly, the coefficients of $e_m \otimes e_n, e_m \otimes e_n^*$ and $e_m^* \otimes e_n^*$ are zero, too.

Furthermore, we have

$$\begin{aligned} r_{23} \wedge_\bullet r_{12} &= \sum_{i,j} e_j \otimes e_i \wedge_\bullet e_j^* \otimes e_i^* = \sum_{i,j} e_j \otimes [-R_{\swarrow}^*(e_i)e_j^* + L_{\succ}^*(e_j^*)e_i] \otimes e_i^* \\ &= \sum_{i,j,k} -e_j \otimes e_k^* \langle e_j^*, e_k \swarrow e_i \rangle \otimes e_i^* + e_j \otimes e_k \langle e_i, e_j^* \succ_\bullet e_k^* \rangle \otimes e_i^* \\ &= \sum_{i,k} -e_k \swarrow e_i \otimes e_k^* \otimes e_i^* + e_i \otimes e_k \otimes e_i^* \succ_\bullet e_k^* = -r_{12} \swarrow_\bullet r_{13} + r_{13} \succ_\bullet r_{23}. \end{aligned}$$

So $Q_1^1 = 0$. Similarly, r satisfies Eqs. (6.15)-(6.19). So the cooperations

$$\alpha_{\mathcal{QD}}(x) = (-1 \otimes L_{\searrow}(x) + R_{\star}(x) \otimes 1)r, \beta_{\mathcal{QD}}(x) = (1 \otimes L_{\vee}(x) - R_{\prec}(x) \otimes 1)r,$$

$$\tilde{\alpha}_{\mathcal{QD}}(x) = (1 \otimes L_{\succ}(x) - R_{\wedge}(x) \otimes 1)r, \tilde{\beta}_{\mathcal{QD}}(x) = (-1 \otimes L_{\star}(x) + R_{\nwarrow}(x) \otimes 1)r$$

induce a quadri-bialgebra structure on $A \oplus A^*$.

On the other hand, for $e_i \in A$, we have:

$$\begin{aligned} \alpha_{\mathcal{QD}}(e_i) &= \sum_{j,k} \{-e_j \otimes (\langle e_j^*, e_k \star e_i \rangle e_k^* + \langle e_j^* \nwarrow_\bullet e_k^*, e_i \rangle e_k) + e_j \star e_i \otimes e_j^*\} \\ &= -\sum_{j,k} \langle e_j^* \nwarrow_\bullet e_k^*, e_i \rangle e_j \otimes e_k = -\alpha(e_i). \end{aligned}$$

Similarly, we have $\beta_{\mathcal{QD}}(e_i) = -\beta(e_i), \tilde{\alpha}_{\mathcal{QD}}(e_i) = -\tilde{\alpha}(e_i)$ and $\tilde{\beta}_{\mathcal{QD}}(e_i) = -\tilde{\beta}(e_i)$. So i_1 is a homomorphism of quadri-bialgebras. Similarly, we prove that i_2 is also a homomorphism of quadri-bialgebras. \square

Definition 6.8. Let $(A, \nwarrow, \nearrow, \swarrow, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ be a quadri-bialgebra. With the quadri-bialgebra structure given in Theorem 6.7, $A \oplus A^*$ is called a *Drinfeld Q-double* of A and we

denote it by $\mathcal{QD}(A)$. On the other hand, due to the symmetry between A and A^* (Remark 5.5), $\tilde{r} := \sum_i e_i^* \otimes e_i$ also induces a (coboundary) quadri-bialgebra structure on $A \oplus A^*$, and we denote it by $\tilde{\mathcal{QD}}(A)$.

Proposition 6.9. *Let $(A, \lrcorner, \rhd, \lhd, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ be a quadri-bialgebra. Suppose that $\alpha, \beta, \tilde{\alpha}$ and $\tilde{\beta}$ are defined by Eqs. (6.1)-(6.4) with $r_{\lrcorner} = r_{\rhd} = r_{\lhd} = r_{\searrow} = r$ respectively.*

(1) *If r satisfies Eqs. (6.14)-(6.19), then T_r is a homomorphism of quadri-bialgebras which is from the quadri-bialgebra given in Remark 5.5 to $(A, \lrcorner, \rhd, \lhd, \searrow, -\alpha, -\beta, -\tilde{\alpha}, -\tilde{\beta})$.*

(2) *If r satisfies Eqs. (6.14)-(6.19) and r is skew-symmetric, then*

$$(6.29) \quad \tilde{T}_r(x + a^*) := x + T_r(a^*), \forall x \in A, a^* \in A^*$$

is a homomorphism of quadri-bialgebras which is from $\tilde{\mathcal{QD}}(A)$ to $(A, \lrcorner, \rhd, \lhd, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$.

Proof. (1) Note that $(1 \otimes \alpha)r = r_{12} \lrcorner r_{13}$, $(\alpha \otimes 1)r = -r_{13} \lrcorner r_{23}$. Set $\lrcorner_* := \alpha^*$. Then for any $a^*, b^* \in A^*$ we have

$$\begin{aligned} T_r(a^* \lrcorner_* b^*) &= \langle 1 \otimes (a^* \lrcorner_* b^*), r \rangle = \langle 1 \otimes a^* \otimes b^*, (1 \otimes \alpha)r \rangle \\ &= \langle 1 \otimes a^* \otimes b^*, r_{12} \lrcorner r_{13} \rangle = T_r(a^*) \lrcorner T_r(b^*) \\ (T_r \otimes T_r)\gamma(a^*) &= T_r(a_{(1)}^*) \otimes T_r(a_{(2)}^*) = \sum_i u_i \otimes u_j \langle a_{(1)}^*, v_i \rangle \langle a_{(2)}^*, v_j \rangle = \langle 1 \otimes 1 \otimes a^*, r_{13} \lrcorner r_{23} \rangle \\ &= -(1 \otimes 1 \otimes a^*)(\alpha \otimes 1)r = -\alpha(T_r(a^*)). \end{aligned}$$

Here we use the Sweedler's notation: $\gamma(a^*) = a_{(1)}^* \otimes a_{(2)}^*$. Similarly T_r also preserves other operations and cooperations. So the conclusion holds.

(2) We still denote the products of $\tilde{\mathcal{QD}}(A)$ by $\lrcorner, \rhd, \lhd, \searrow$. First we prove that \tilde{T}_r is a homomorphism of quadri-algebras, that is, $\tilde{T}_r(\mu \diamond \nu) = \tilde{T}_r(\mu) \diamond \tilde{T}_r(\nu)$ for any $\mu, \nu \in A \oplus A^*$ and $\diamond \in \{\lrcorner, \rhd, \lhd, \searrow\}$. It is obvious when $\mu, \nu \in A$. Moreover, for any $x \in A, a^* \in A^*$ we have

$$\begin{aligned} \tilde{T}_r(x \lrcorner a^*) &= \tilde{T}_r(R_{\searrow}^*(x)a^* + L_{\star}^*(a^*)x) = T_r(R_{\searrow}^*(x)a^*) + L_{\star}^*(a^*)x \\ &= (1 \otimes a^*)((1 \otimes R_{\searrow}(x))r) + (a^* \otimes 1)((-1 \otimes L_{\lrcorner}(x) + R_{\searrow}(x) \otimes 1)r) \\ &= (1 \otimes a^*)((L_{\lrcorner}(x) \otimes 1)r) = \tilde{T}_r(x) \lrcorner \tilde{T}_r(a^*), \end{aligned}$$

where we use the fact that r is skew-symmetric. Similarly, $\tilde{T}_r(a^* \lrcorner x) = \tilde{T}_r(a^*) \lrcorner \tilde{T}_r(x)$ and when $\diamond \in \{\rhd, \lhd, \searrow\}$, $\tilde{T}_r(\mu \diamond \nu) = \tilde{T}_r(\mu) \diamond \tilde{T}_r(\nu)$ for all $\mu \in A, \nu \in A^*$ or $\mu \in A^*, \nu \in A$. On the other hand, by (1), for any $a^*, b^* \in A^*$, we have

$$\tilde{T}_r(a^* \diamond b^*) = T_r(a^* \diamond b^*) = T_r(a^*) \diamond T_r(b^*) = \tilde{T}_r(a^*) \diamond \tilde{T}_r(b^*),$$

where $\diamond \in \{\lrcorner, \rhd, \lhd, \searrow\}$. Therefore, \tilde{T}_r is a homomorphism of quadri-algebras.

Furthermore, let $\{e_1, \dots, e_n\}$ be a basis of A and let $\{e_1^*, \dots, e_n^*\}$ be the dual basis. Define $\tilde{r} := \sum_i e_i^* \otimes e_i$. Then we have

$$(\tilde{T}_r \otimes \tilde{T}_r)(\tilde{r}) = \sum_i \tilde{T}_r(e_i^*) \otimes \tilde{T}_r(e_i) = \sum_i (1 \otimes e_i^*)(r) \otimes e_i = r,$$

which implies \tilde{T}_r is a homomorphism of quadri-coalgebras. So the conclusion follows. \square

7. Q-EQUATION

By Theorem 6.1, we have the following conclusion.

Proposition 7.1. *Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra and $r \in A \otimes A$ be skew-symmetric. Then the comultiplications $\alpha, \beta, \tilde{\alpha}$ and $\tilde{\beta}$ defined by Eqs. (6.1)-(6.4) with $r_{\nwarrow} = r_{\nearrow} = r_{\swarrow} = r_{\searrow} = r$ respectively make $(A, \nwarrow, \nearrow, \swarrow, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ into a quadri-bialgebra if r satisfies Eqs. (6.14)-(6.19).*

Definition 7.2. ([Bai3]) Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra and let $(V, l_{\nwarrow}, r_{\nwarrow}, l_{\nearrow}, r_{\nearrow}, l_{\swarrow}, r_{\swarrow}, l_{\searrow}, r_{\searrow})$ be a bimodule. An \mathcal{O} -operator of A associated to the bimodule V is a linear map T from V to A such that for all $u, v \in V$,

$$(7.1) \quad T(u) \circ T(v) = T(l_{\circ}(T(u))v + r_{\circ}(T(v))u), \circ \in \{\nwarrow, \nearrow, \swarrow, \searrow\}.$$

Proposition 7.3. ([Bai3]) *Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra and let $r \in A \otimes A$ be skew-symmetric. Then the following conditions are equivalent:*

- (1) *r satisfies Eqs. (6.14)-(6.15);*
- (2) *r satisfies Eqs. (6.16)-(6.17);*
- (3) *r satisfies Eqs. (6.18)-(6.19);*
- (4) *T_r is an \mathcal{O} -operator of A associated to the bimodule $(A^*, R_{\nwarrow}^*, L_{\star}^*, -R_{\swarrow}^*, -L_{\swarrow}^*, -R_{\nearrow}^*, -L_{\nearrow}^*, R_{\searrow}^*, L_{\star}^*)$.*
- (5) *T_r is an \mathcal{O} -operator of A_v associated to the bimodule $(A^*, -R_{\swarrow}^*, L_{\nearrow}^*, R_{\searrow}^*, -L_{\nearrow}^*)$;*

Definition 7.4. ([Bai3]) Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra and let $r \in A \otimes A$. A set of equations (6.14) and (6.15) is called *Q-equation* in $(A, \nwarrow, \nearrow, \swarrow, \searrow)$.

Remark 7.5. In the sense of Proposition 7.3 (in terms of \mathcal{O} -operators), *Q-equation* in a quadri-algebra can be regarded as an analogue of the classical Yang-Baxter equation in a Lie algebra ([Bai1, K]), which led to the introduction of *Q-equation* in [Bai3].

The following conclusion is obvious:

Corollary 7.6. *Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra and let $r \in A \otimes A$ be skew-symmetric. Then the comultiplications $\alpha, \beta, \tilde{\alpha}$ and $\tilde{\beta}$ defined by Eqs. (6.1)-(6.4) with $r_{\nwarrow} = r_{\nearrow} = r_{\swarrow} = r_{\searrow} = r$ respectively make $(A, \nwarrow, \nearrow, \swarrow, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ into a quadri-bialgebra if r is a solution of *Q-equation*.*

Proposition 7.7. *Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra and let r be a skew-symmetric solution of *Q-equation*. Then the quadri-algebra structure $(\nwarrow_{\bullet}, \nearrow_{\bullet}, \swarrow_{\bullet}, \searrow_{\bullet})$ on the Drinfeld *Q-double**

$\mathcal{QD}(A)$ can be given as follows (for any $x \in A, a^*, b^* \in A^*$):

$$(7.2) \quad a^* \nwarrow_{\bullet} b^* = R_{\searrow}^*(T_r(a^*))b^* + L_{\star}^*(T_r(b^*))a^*, a^* \nearrow_{\bullet} b^* = -R_{\swarrow}^*(T_r(a^*))b^* - L_{\prec}^*(T_r(b^*))a^*,$$

$$(7.3) \quad a^* \swarrow_{\bullet} b^* = -R_{\searrow}^*(T_r(a^*))b^* - L_{\wedge}^*(T_r(b^*))a^*, a^* \searrow_{\bullet} b^* = R_{\star}^*(T_r(a^*))b^* + L_{\searrow}^*(T_r(b^*))a^*,$$

$$(7.4) \quad a^* \nwarrow_{\bullet} x = -T_r(L_{\star}^*(x)a^*) + T_r(a^*) \nwarrow x + L_{\star}^*(x)a^*,$$

$$(7.5) \quad a^* \nearrow_{\bullet} x = T_r(L_{\prec}^*(x)a^*) + T_r(a^*) \nearrow x - L_{\prec}^*(x)a^*,$$

$$(7.6) \quad a^* \swarrow_{\bullet} x = T_r(L_{\wedge}^*(x)a^*) + T_r(a^*) \swarrow x - L_{\wedge}^*(x)a^*,$$

$$(7.7) \quad a^* \searrow_{\bullet} x = -T_r(L_{\searrow}^*(x)a^*) + T_r(a^*) \searrow x + L_{\searrow}^*(x)a^*,$$

$$(7.8) \quad x \nwarrow_{\bullet} a^* = R_{\searrow}^*(x)a^* + x \nwarrow T_r(a^*) - T_r(R_{\searrow}^*(x)a^*),$$

$$(7.9) \quad x \nearrow_{\bullet} a^* = -R_{\swarrow}^*(x)a^* + x \nearrow T_r(a^*) + T_r(R_{\swarrow}^*(x)a^*),$$

$$(7.10) \quad x \swarrow_{\bullet} a^* = -R_{\searrow}^*(x)a^* + x \swarrow T_r(a^*) + T_r(R_{\searrow}^*(x)a^*),$$

$$(7.11) \quad x \searrow_{\bullet} a^* = R_{\star}^*(x)a^* + x \searrow T_r(a^*) - T_r(R_{\star}^*(x)a^*).$$

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of A and let $\{e_1^*, \dots, e_n^*\}$ be its dual basis. Suppose that

$$e_i \nwarrow e_j = \sum_{i,j} c_{ij}^k e_k, e_i \nearrow e_j = \sum_{i,j} d_{ij}^k e_k, e_i \swarrow e_j = \sum_{i,j} \tilde{c}_{ij}^k e_k, e_i \searrow e_j = \sum_{i,j} \tilde{d}_{ij}^k e_k,$$

and $r = \sum_{i,j} a_{ij} e_i \otimes e_j$, where $a_{ij} = -a_{ji}$. Then $T_r(e_i^*) = \sum_k a_{ki} e_k$. For any k, l we have

$$\begin{aligned} e_k^* \nwarrow_{\bullet} e_l^* &= \sum_s \langle e_k^* \otimes e_l^*, \alpha(e_s) \rangle e_s^* \\ &= \sum_{t,s} [-a_{kt} \tilde{d}_{st}^l + a_{tl} (c_{ts}^k + d_{ts}^k + \tilde{c}_{ts}^k + \tilde{d}_{ts}^k)] e_s^* = R_{\searrow}^*(T_r(e_k^*))e_l^* + L_{\star}^*(T_r(e_l^*))e_k^*. \end{aligned}$$

Hence for any $a^*, b^* \in A^*$ we have that $a^* \nwarrow_{\bullet} b^* = R_{\searrow}^*(T_r(a^*))b^* + L_{\star}^*(T_r(b^*))a^*$. Similarly $a^* \nearrow_{\bullet} b^* = -R_{\swarrow}^*(T_r(a^*))b^* - L_{\prec}^*(T_r(b^*))a^*$. So Eq. (7.2) holds. Eq. (7.3) is proved in a similar way. On the other hand, we have

$$\begin{aligned} R_{\searrow}^*(e_k^*)e_l &= \sum_s \langle e_l, e_s^* \searrow_{\bullet} e_k^* \rangle e_s = \sum_s \langle e_l, R_{\star}^*(T_r(e_s^*))e_k^* + L_{\searrow}^*(T_r(e_k^*))e_s^* \rangle e_s \\ &= \sum_s \langle e_s^*, -T_r(L_{\star}^*(e_l)e_k^*) + T_r(e_k^*) \nwarrow_{\bullet} e_l \rangle e_s = -T_r(L_{\star}^*(e_l)e_k^*) + T_r(e_k^*) \nwarrow_{\bullet} e_l. \end{aligned}$$

Thus, $e_k^* \nwarrow_{\bullet} e_l = R_{\searrow}^*(e_k^*)e_l + L_{\star}^*(e_l)e_k^* = -T_r(L_{\star}^*(e_l)e_k^*) + T_r(e_k^*) \nwarrow_{\bullet} e_l + L_{\star}^*(e_l)e_k^*$. Therefore Eq. (7.4) holds. Eqs. (7.5)-(7.11) are verified in a same way. \square

Proposition 7.8. ([Bai3]) *Let $(A, \lrcorner, \nearrow, \swarrow, \searrow)$ be a quadri-algebra and let $r \in A \otimes A$. Suppose that r is skew-symmetric and nondegenerate. Then r is a solution of Q -equation if and only if the inverse of the isomorphism $A^* \rightarrow A$ induced by r , regarded as a bilinear form ω on A , satisfies*

$$(7.12) \quad \omega(x, y \wedge z) = -\omega(x \swarrow y, z) + \omega(z \succ x, y),$$

$$(7.13) \quad \omega(x, y \vee z) = \omega(x \prec y, z) - \omega(z \nearrow x, y), \quad \forall x, y, z \in A.$$

Definition 7.9. Let $(A, \lrcorner, \nearrow, \swarrow, \searrow)$ be a quadri-algebra. A bilinear form $\omega : A \otimes A \rightarrow \mathbb{F}$ is called a 2-cocycle of A if it satisfies Eqs. (7.12) and (7.13).

It is easy to show that if $\omega : A \otimes A \rightarrow \mathbb{F}$ is a 2-cocycle of A , then for any $x, y \in A$, $\mathfrak{B}(x, y) := \omega(x, y) + \omega(y, x)$ is a 2-cocycle on A_v .

Proposition 7.10. *Let $(A, \lrcorner, \nearrow, \swarrow, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ be a quadri-bialgebra obtained from a skew-symmetric solution of Q -equation. Let $(A, A^*, R_{\searrow}^*, L_{\star}^*, -R_{\vee}^*, -L_{\prec}^*, -R_{\succ}^*, -L_{\wedge}^*, R_{\star}^*, L_{\prec}^*, R_{\searrow}^*, L_{\star}^*, -R_{\vee}^*, -L_{\prec}^*, -R_{\succ}^*, -L_{\wedge}^*, R_{\star}^*, L_{\prec}^*, R_{\searrow}^*)$ be the corresponding matched pair of quadri-algebras, where the subscript $*$ denotes the quadri-algebra structure on A^* . Set*

$$(7.14) \quad A \ltimes A^* =: A \ltimes_{R_{\searrow}^*, L_{\star}^*, -R_{\vee}^*, -L_{\prec}^*, -R_{\succ}^*, -L_{\wedge}^*, R_{\star}^*, L_{\prec}^*} A^*.$$

(1) $A \ltimes_{R_{\searrow}^*, L_{\star}^*, -R_{\vee}^*, -L_{\prec}^*, -R_{\succ}^*, -L_{\wedge}^*, R_{\star}^*, L_{\prec}^*} A^*$ is isomorphic to $A \ltimes A^*$ as quadri-algebras.

(2) The skew-symmetric solutions of Q -equation in A are in one-to-one correspondence with linear maps $T_r : A^* \rightarrow A$ whose graphs are Lagrangian sub-quadri-algebras of $A \ltimes A^*$ with respect to the bilinear form (3.2). Here the graph of a linear map $T : A^* \rightarrow A$ is defined as $\text{graph}(T) := \{(T(a^*), a^*) | a^* \in A^*\} \subset A \ltimes A^*$.

Proof. We denote the quadri-algebra structure on $A \ltimes A^*$ by $\lrcorner_{\dagger}, \nearrow_{\dagger}, \swarrow_{\dagger}, \searrow_{\dagger}$. Define a linear map $\theta : A \ltimes_{R_{\searrow}^*, L_{\star}^*, -R_{\vee}^*, -L_{\prec}^*, -R_{\succ}^*, -L_{\wedge}^*, R_{\star}^*, L_{\prec}^*} A^* \rightarrow A \ltimes A^*$ by

$$\theta(x, a^*) := (T_r(a^*) + x, a^*), \quad \forall x \in A; a^* \in A^*.$$

It is straightforward to check that θ is a homomorphism of quadri-algebras. Moreover, θ is bijective. So (1) holds.

Suppose that r is a skew-symmetric solution of Q -equation. Then by (1) we know that $\theta(A^*) = \text{graph}(T_r)$ and $\theta(A) = A$ are isotropic complementary sub-quadri-algebras of $A \ltimes A^*$ and dual to each others with respect to the bilinear form (3.2). So $\text{graph}(T_r)$ is a Lagrangian sub-quadri-algebra of $A \ltimes A^*$. Conversely, let $T_r : A^* \rightarrow A$ be a linear map whose graph $\text{graph}(T_r)$ is a Lagrangian sub-quadri-algebra of $A \ltimes A^*$. Since $\text{graph}(T_r)$ is Lagrangian, r is skew-symmetric. Moreover, since $\text{graph}(T_r)$ is a sub-quadri-algebra of $A \ltimes A^*$, we have

$$\begin{aligned} (T_r(a^*), a^*) \lrcorner_{\dagger} (T_r(b^*), b^*) &= (T_r(a^*) \lrcorner T_r(b^*), R_{\searrow}^*(T_r(a^*))b^* + L_{\star}^*(T_r(b^*))a^*) \\ &= (T_r(R_{\searrow}^*(T_r(a^*))b^* + L_{\star}^*(T_r(b^*))a^*), R_{\searrow}^*(T_r(a^*))b^* + L_{\star}^*(T_r(b^*))a^*). \end{aligned}$$

Therefore, $T_r(a^*) \nwarrow T_r(b^*) = T_r(R_{\nwarrow}^*(T_r(a^*))b^* + L_{\star}^*(T_r(b^*))a^*)$. Similarly we have

$$\begin{aligned} T_r(a^*) \nearrow T_r(b^*) &= T_r(-R_{\nearrow}^*(T_r(a^*))b^* - L_{\nwarrow}^*(T_r(b^*))a^*), \\ T_r(a^*) \swarrow T_r(b^*) &= T_r(-R_{\swarrow}^*(T_r(a^*))b^* - L_{\searrow}^*(T_r(b^*))a^*), \\ T_r(a^*) \searrow T_r(b^*) &= T_r(R_{\searrow}^*(T_r(a^*))b^* + L_{\swarrow}^*(T_r(b^*))a^*). \end{aligned}$$

So by Proposition 7.3, we know that r is a skew-symmetric solution of Q -equation. \square

8. CONSTRUCTION OF LINEAR OPERATORS ON SOME “DOUBLE SPACES” OF QUADRI-ALGEBRAS

An interesting (and important) feature of dendriform dialgebras, quadri-algebras and other similar algebra structures mentioned in [EG1] is that they have close relations with various linear operators in combinatorics ([Ag1, Ag2, Ag3, AL, Bax, E1, E2, EG1, R1, R2]).

Definition 8.1. Let A be a vector space with a set of bilinear operations $\Omega := \{*_n : A \otimes A \rightarrow A, n = 1, \dots, m\}$. A linear operator P on A is called a *Rota-Baxter operator of weight λ* ($\in \mathbb{F}$) if, for each $*$ $\in \Omega$, we have

$$(8.1) \quad P(x) * P(y) = P(P(x) * y + x * P(y) + \lambda x * y), \forall x, y \in A.$$

A linear operator N on A is called a *Nijenhuis operator* if, for each $*$ $\in \Omega$, we have

$$(8.2) \quad N(x) * N(y) = N(N(x) * y + x * N(y) - N(x * y)), \forall x, y \in A.$$

We have the following relationship between Rota-Baxter operators of weight λ and Nijenhuis operators ([E1]).

Proposition 8.2. *With the notations in Definition 8.1, if $N : A \rightarrow A$ is a Nijenhuis operator on A satisfying $N^2 = \lambda^2 \text{id}$, then $P := \frac{-\lambda \text{id} - N}{2}$ is a Rota-Baxter operator of weight λ (on A).*

Proof. It is straightforward. \square

Proposition 8.3. *Let $(A, \nwarrow, \nearrow, \swarrow, \searrow, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ be a quadri-bialgebra, which is induced from a skew-symmetric solution r of Q -equation. If, in addition, r is nondegenerate, then for all $x \in A, a^* \in A^*$,*

$$(8.3) \quad N_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}((x, a^*)) := (\lambda_1 T_r(a^*) + \lambda_2 x, \lambda_3 T_r^{-1}(x) + \lambda_4 a^*)$$

is a Nijenhuis operator on the Drinfeld Q -double $\mathcal{QD}(A)$, where $\lambda_i \in \mathbb{F}$, $i = 1, 2, 3, 4$.

Proof. It is straightforward. \square

In the following we assume the coefficient field \mathbb{F} is the real field \mathbb{R} and let $\lambda \in \mathbb{R}$. With the conditions in Proposition 8.3, if we consider the Nijenhuis operators satisfying $N_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}^2 = \lambda^2 \text{id}$

and apply Proposition 8.2, then we can get three families of Rota-Baxter operators of weight λ on $\mathcal{QD}(A)$:

$$\begin{aligned}
(\text{F1}) \quad & P_{\lambda,k}^+((x, a^*)) := \frac{-\lambda - N_{0,\lambda,k,-\lambda}}{2}((x, a^*)) = (-\lambda x, -\frac{k}{2}T_r^{-1}(x)) \\
& \text{or } P_{\lambda,k}^-((x, a^*)) := \frac{-\lambda - N_{0,-\lambda,k,\lambda}}{2}((x, a^*)) = (0, -\frac{k}{2}T_r^{-1}(x) - \lambda a^*), k \neq 0; \\
(\text{F2}) \quad & \hat{P}_{\lambda,\hat{k}}^+((x, a^*)) := \frac{-\lambda - N_{\hat{k},\lambda,0,-\lambda}}{2}((x, a^*)) = (-\frac{\hat{k}}{2}T_r(a^*) - \lambda x, 0) \\
& \text{or } \hat{P}_{\lambda,\hat{k}}^-((x, a^*)) := \frac{-\lambda - N_{\hat{k},-\lambda,0,\lambda}}{2}((x, a^*)) = (-\frac{\hat{k}}{2}T_r(a^*), -\lambda a^*), (\hat{k}, \lambda) \neq (0, 0); \\
(\text{F3}) \quad & P_{\lambda,k_1,k_2}((x, a^*)) := \frac{-\lambda - N_{k_1,k_2,\frac{\lambda^2-k_2^2}{k_1},-k_2}}{2}((x, a^*)) \\
& = (-\frac{k_1}{2}T_r(a^*) - \frac{(k_2 + \lambda)}{2}x, \frac{(k_2^2 - \lambda^2)}{2k_1}T_r^{-1}(x) + \frac{(k_2 - \lambda)}{2}a^*), k_2 \neq \pm\lambda, k_1 \neq 0,
\end{aligned}$$

for any $x \in A, a^* \in A^*$, where $k, \hat{k}, k_1, k_2 \in \mathbb{R}$. Here we exclude the trivial cases that $P = -\lambda \text{id}$. Furthermore, it is easy to check that these Rota-Baxter operators are idempotents (i.e., $P^2 = P$) if and only if $\lambda = -1$, i.e., they are Rota-Baxter operators of weight -1 . We write down them explicitly as follows (see the discussion at the end of this section for their important roles in renormalization in quantum field theory):

$$\begin{aligned}
(\text{G1}) \quad & P_{-1,k}^+((x, a^*)) := \frac{1 - N_{0,-1,k,1}}{2}((x, a^*)) = (x, -\frac{k}{2}T_r^{-1}(x)) \\
& \text{or } P_{-1,k}^-((x, a^*)) := \frac{1 - N_{0,1,k,-1}}{2}((x, a^*)) = (0, -\frac{k}{2}T_r^{-1}(x) + a^*), k \neq 0; \\
(\text{G2}) \quad & \hat{P}_{-1,\hat{k}}^+((x, a^*)) := \frac{1 - N_{\hat{k},-1,0,1}}{2}((x, a^*)) = (-\frac{\hat{k}}{2}T_r(a^*) + x, 0) \\
& \text{or } \hat{P}_{-1,\hat{k}}^-((x, a^*)) := \frac{1 - N_{\hat{k},1,0,-1}}{2}((x, a^*)) = (-\frac{\hat{k}}{2}T_r(a^*), a^*); \\
(\text{G3}) \quad & P_{-1,k_1,k_2}((x, a^*)) := \frac{1 - N_{k_1,k_2,\frac{1-k_2^2}{k_1},-k_2}}{2}((x, a^*)) \\
& = (-\frac{k_1}{2}T_r(a^*) - \frac{(k_2 - 1)}{2}x, \frac{(k_2^2 - 1)}{2k_1}T_r^{-1}(x) + \frac{(k_2 + 1)}{2}a^*), k_2 \neq \pm 1, k_1 \neq 0,
\end{aligned}$$

for all $x \in A, a^* \in A^*$, where $k, \hat{k}, k_1, k_2 \in \mathbb{R}$.

On the other hand, the requirement that r is nondegenerate can be dropped if λ_3 appearing in the definition of $N_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}$ is equal to zero. More precisely, we have the following conclusion.

Proposition 8.4. *Let $(A, \curvearrowright, \curvearrowleft, \curvearrowright, \curvearrowleft, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ be a quadri-bialgebra, which is induced from a skew-symmetric solution r of Q -equation. Then, for all $x \in A, a^* \in A^*$,*

$$(8.4) \quad N_{\lambda_1,\lambda_2,\lambda_3}((x, a^*)) := (\lambda_1 T_r(a^*) + \lambda_2 x, \lambda_3 a^*)$$

is a Nijenhuis operator on $\mathcal{QD}(A)$, where $\lambda_i \in \mathbb{F}$, $i = 1, 2, 3$.

Proof. It is straightforward. □

Remark 8.5. Similarly, we consider the case that $N_{\lambda_1, \lambda_2, \lambda_3} = \lambda^2 \text{id}$ for $\lambda \in \mathbb{R}$ and then apply Proposition 8.2 to get certain families of Rota-Baxter operators (of weight λ) on $\mathcal{QD}(A)$. In fact, one can show that the Rota-Baxter operators (of weight λ) are given by (F2) and the Rota-Baxter operators which are idempotents are given by (G2), where $x \in A, a^* \in A^*$.

Proposition 8.6. *Let $(A, \nwarrow, \nearrow, \swarrow, \searrow)$ be a quadri-algebra, then*

$$(8.5) \quad N_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}((x, y)) := (\lambda_1 y + \lambda_2 x, \lambda_3 x + \lambda_4 y)$$

is a Nijenhuis operator on $A_v \ltimes_{L \nearrow, R \nwarrow, L \swarrow, R \searrow} A$, where $x, y \in A$ and $\lambda_i \in \mathbb{F}$, $i = 1, 2, 3, 4$. Moreover, if $(A_v, A_v, L \nearrow, R \nwarrow, L \swarrow, R \searrow, L \nearrow, R \nwarrow, L \swarrow, R \searrow)$ is a matched pair of dendriform dialgebras, then the linear operator defined by Eq. (8.5) is also a Nijenhuis operator on $A_v \bowtie_{L \nearrow, R \nwarrow, L \swarrow, R \searrow}^{L \nearrow, R \nwarrow, L \swarrow, R \searrow} A_v$. On the other hand,

$$(8.6) \quad \theta((x, y)) := (y + x, x)$$

is an isomorphism of dendriform dialgebras from $A_v \bowtie_{L \nearrow, R \nwarrow, L \swarrow, R \searrow}^{L \nearrow, R \nwarrow, L \swarrow, R \searrow} A_v$ to $A_v \ltimes_{L \nearrow, R \nwarrow, L \swarrow, R \searrow} A$.

Proof. It is straightforward. □

Remark 8.7. Similarly, we can also consider the case that $N_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} = \lambda^2 \text{id}$ and then apply Proposition 8.2 to get certain families of Rota-Baxter operators of weight λ on the “double spaces” given in Proposition 8.6.

Remark 8.8. One can use these Nijenhuis operators and Rota-Baxter operators (of weight λ) to construct *NS-algebra* and *dendriform trialgebra* structures (see [EG1] and the references therein). Moreover, it is easy to show that these Nijenhuis operators also satisfy Eq. (8.2) on their corresponding associative algebras, that is, they are the so-called *associative Nijenhuis tensors* of the associated associative algebras in the sense of [CGM], where this notion was introduced in the study of Wigner problem in quantum physics.

Remark 8.9. Obviously, if P is a Rota-Baxter operator of weight λ on a quadri-algebra, then it is a Rota-Baxter operator of weight λ on its associated associative algebra. Furthermore, Rota-Baxter operators on associative algebras which are idempotents play a key role in the *algebraic Birkhoff decomposition* in pQFT ([EGK1, EGK2], also see [EG2] for a good survey of this topic). As we have discussed, we have constructed certain families of Rota-Baxter operators on the “double spaces” of quadri-algebras.

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